DOCUMENT RESUME

ED 021 734 SE 004 569

HANDBOOK FOR GENERAL MATHEMATICS.

Maryland State Dept. of Education, Baltimore.

Pub Date 66

Note- 147p.

EDRS Price MF-$0.75 HC-$5.96

Descriptors- ALGEBRA, ARITHMETIC, *COURSE CONTENT, *CURRICULUM, GEOMETRY, INSTRUCTIONAL MATERIALS, *MATHEMATICS, NONCOLLEGE PREPARATORY STUDENTS, *SECONDARY SCHOOL MATHEMATICS, *TEACHING GUIDES

Identifiers- Mary and State Department of Education

This handbook for general mathematics presents some classroom materials to aid the teacher of non college-bound pupils. The materials are intended to be of assistance to teachers in teaching the materials in a manner which will maximize the mathematical flexibility as well as the mathematical skills of the pupils. The goal was to prepare a booklet of ideas and approaches to mathematics for all teachers of general mathematics. Classroom materials are provided in the content areas of natural numbers, integers, rational numbers, and geometry. (RP)

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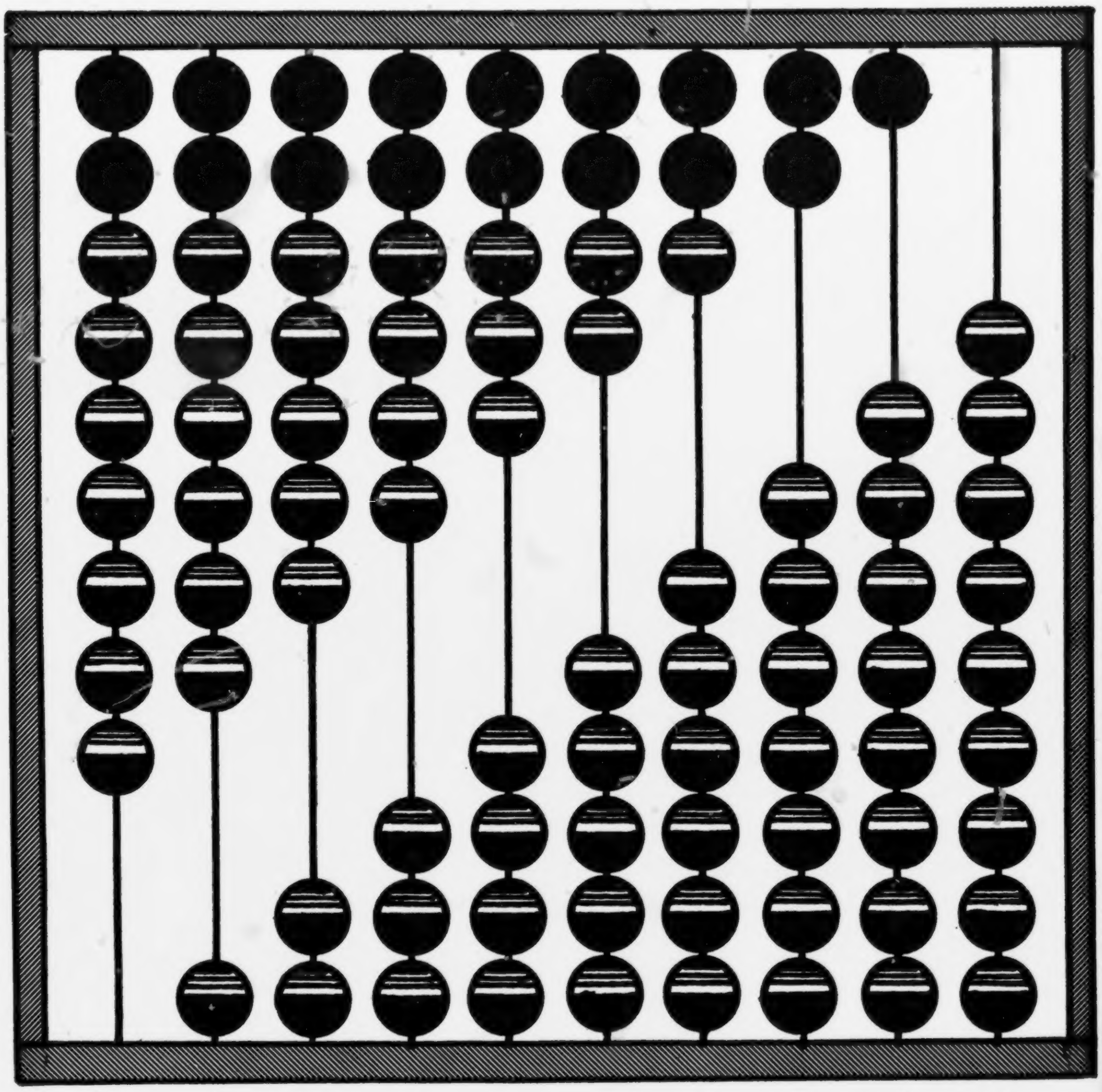
HANDBOOK FOR GENERAL
MATHEMATICS



MARYLAND

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HANDBOOK FOR

GENERAL MATHEMATICS

FOREWORD

Education in Maryland public schools is based upon the premise that we have a responsibility to develop in all youth a common educational literacy. Mathematics is a basic subject in this general educational heritage. Mathematics has much to contribute to an understanding of a world which is changing constantly. Scientific, industrial, and sociological change is the rule of the day. Whole categories of jobs disappear and entirely new types of jobs appear. Mathematics courses need to provide youth with a broad base of mathematical abilities to enable them to adapt to this world of change.

Many of today's young people who do not go on to college limit their mathematical training to those courses which are termed "general mathematics." For the most part these youths are least equipped to face the changing world. It is imperative that these courses be taught in a manner which will maximize the mathematical flexibility as well as the mathematical skill of the pupils.

It is, therefore, most appropriate that efforts be increased to improve the content and instruction in general mathematics. We are very happy to provide this Handbook as one step toward encouraging such improvement.

James A. Sensenbaugh

State Superintendent of Schools

TABLE OF CONTENTS

Chapter Page

FOREWORD . . . .

PREFACE ii

INTRODUCTION 1

I. POINT OF VIEW 18

II. NATURAL NUMBERS 23

III. INTEGERS 38

IV. RATIONAL NUMBERS 55

V. MISCELLANEOUS TOPICS 74

BIBLIOGRAPHY 137

SUGGESTED IMPROVEMENTS . . . .

PREFACE

During the summer of 1966 a committee of Maryland teachers met at the invitation of the State Supervisor of Mathematics. In response to suggestions made by many local supervisors, the committee was asked to prepare some materials which would aid the teacher of non college-bound pupils. This publication is the result of the effort put forth by that committee. The industry and efficiency of the committee can easily be seen in the comprehensiveness and quality of the material contained herein. This handbook was written with the hope that it would be of interest and of some assistance to many teachers in every school system in the State of Maryland.

It is appropriate that we acknowledge the following people who have contributed so much to this publication:

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Mrs. Floretta Fyhr, Mathematics Department Chairman, Northern High School, Baltimore City Public Schools

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Mr. William Gerardi, Mathematics Supervisor, Baltimore City Public Schools.

A note of thanks is also extended to supervisors, teachers, and others who contributed both formally and informally during the preparation of this publication and to the Howard County Public Schools for the use of facilities in their Board of Education.

INTRODUCTION

Exponents by Discovery

Repeat										
Total	1	2	3	4	5	6	7	8	9	10

Duplicate enough of this type of card so that each student will have several in his possession. These cards can represent order forms received from the purchasing department by the shipping department. The shipping clerk at a factory receives 3 of these cards from the purchasing department, marked as shown.

A

Repeat										
			x							
			x							
Total	1	2	3	4	5	6	7	8	9	10

B

Repeat										
					x x					
Total	1	2	3	4	5	6	7	8	9	10

C

Repeat										
						x x				
Total	1	2	3	4	5	6	7	8	9	10

He immediately marks "9" in the space under the word total on card A, "25" on card B, "36" on card C. He hands the cards to a stockboy who has no difficulty in filling the order because he simply looks at the number in the space under total. Sam is a very ambitious stockboy and would like to be promoted as quickly as possible. However, one requirement for promotion is to be able to interpret the cards correctly so that he may fill in the total on the cards when received.

During the shipping clerk's lunchbreak, a rush order came to the stock room. The card looked like this.

Repeat										
		RUSH								
							x			
							x			
Total	1	2	3	4	5	6	7	8	9	10

Can you assist Sam in filling the order?

*If the students cannot help Sam, they should review several other order cards to see if they can discover how the x's over a number influence the total.

This should lead the class to discover that $\begin{pmatrix} x \\ \underline{x} \\ 7 \end{pmatrix}$ indicates $7 \times 7 = 49$. This could be written as $7^2 = 49$ now or later.

If you were Sam and received a card with $\begin{pmatrix} x \\ x \\ \underline{x} \\ 7 \end{pmatrix}$, could you fill the order? Continue this line of questioning by putting $\begin{pmatrix} x \\ x \\ x \\ \underline{x} \\ 2 \end{pmatrix}$ and asking the class for a total. Could you order this same amount another way? $\begin{pmatrix} x \\ \underline{x} \\ 4 \end{pmatrix}$ For a variation in this lesson, prepare cards giving the total and have the class place x's in the correct position.(See figure on next page.)

Repeat										
Total	1	2	3	4	5	6	7	8	9	10
64										

Other suggested uses for these cards:

a)

Repeat										
2			x							
			x							
Total	1	2	3	4	5	6	7	8	9	10
18										

$2(3^2) = 18$

Note: The number in the repeat column indicates a double order, triple order, etc., depending upon number appearing there.

b)

Repeat										
			x							
			x		x					
Total	1	2	3	4	5	6	7	8	9	10
14										

$3^2+5 = 14$

c)

Repeat										
3				x x			x x			
Total	1	2	3	4	5	6	7	8	9	10
195										

$3(4^2+7^2) = 195$

Properties

How about trying this series of problems to motivate a lesson on properties?

Directions: Insert operation symbols in the squares, and use grouping symbols to make each statement true.

Problems:

1) 4 ☐ 4 ☐ 4 = 0
2) 4 ☐ 4 ☐ 4 = 12
3) 4 ☐ 4 ☐ 4 = 5
4) 4 ☐ 4 ☐ 4 = 4
5) 4 ☐ 4 ☐ 4 = 2
6) 4 ☐ 4 ☐ 4 = $\frac{1}{4}$
7) 4 ☐ 4 ☐ 4 = 32
8) 4 ☐ 4 ☐ 4 = $\frac{1}{2}$
9) 4 ☐ 4 ☐ 4 = 64
10) 4 ☐ 4 ☐ 4 = -3

Sample answer:

1) $(4 \boxed{-} 4) \boxed{\times} 4 = 0$

or

$4 \boxed{\times} (4 \boxed{-} 4) = 0$

Use this table to review the operations for integers following the same directions given previously.

1) 8 ☐ 2 ☐ 6 = 12
2) 6 ☐ -3 ☐ 2 = -36
3) 7 ☐ -1 ☐ -10 = -17
4) 12 ☐ 10 ☐ 3 = 6
5) -9 ☐ 2 ☐ -6 = 3
6) 12 ☐ 10 ☐ 3 = -1
7) 8 ☐ 2 ☐ 6 = -4

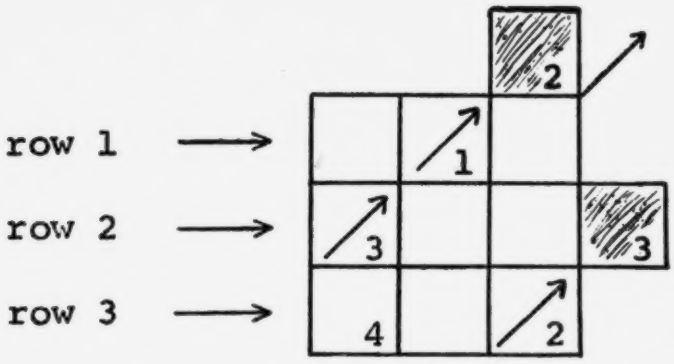
Magic Squares

This is an interesting way to review the concepts of addition and subtraction.

To make a magic square, we must place a different number in each cell in such a way that the sum of every horizontal row of cells, the sum of every vertical column of cells, and the sum of the diagonals will add up to the same number. Magic squares may be made that have an odd number of cells or an even number of cells. Following are the directions for constructing a 9-celled magic square, using the numbers from 1 to 9 inclusive.

Directions:

1. Draw a square with 9 cells.



8	1	6
3	5	7
4	9	2

Magic Square I

2. Place the number 1 in the middle cell of the top row.
3. By moving diagonally up to the right we begin to fill in the other spaces, but this puts us out of the square. When this occurs we simply drop to the last cell in that column or row.
4. Moving diagonally from 3 up to the right, we find ourselves in a space that is already occupied. When this occurs, go back to starting point (3) and put the number 4 in the cell below the 3. This rule also applies if you move out of the square on a diagonal.
5. Continue until all cells are filled.

Ask the class to observe anything unusual resulting from the arrangement of the numerals. It should be apparent to the student that the sum of the numbers of each row is 15;

each column sum is 15; each diagonal sum is 15. The 9-celled magic square is made with 9 consecutive numerals. To change the square, each numeral in Magic Square I may be increased by the same numeral. If each number in Magic Square I is increased by 1, the result is Magic Square II.

9	2	7
4	6	8
5	10	3

Magic Square II

Using this method, squares can be made for class use by omitting numerals in certain cells.

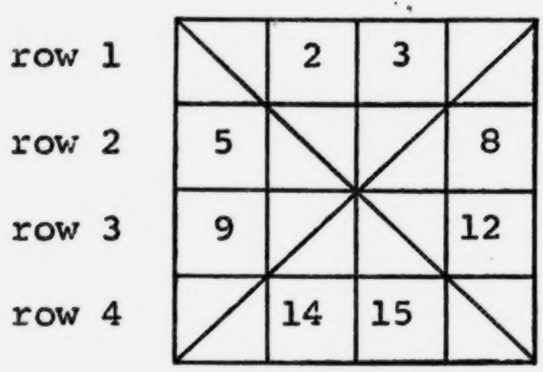
25	▨	23
20	22	24
▨	26	▨

Magic Square III

This square was constructed by increasing each numeral in Magic Square I by 17.

One can make more difficult magic squares using 16 cells, 25 cells, etc.

To construct a magic square containing 16 cells (see below) we do the following:



Magic Square IV

Figure 1

First:

Draw two diagonals in the square. Any cell which contains a diagonal is left without a numeral at this time. Start in the upper left-hand corner and count consecutively across the first row, then across the second row, etc., inserting numerals in the cells which do not contain a diagonal (see Figure 1).

Second:

To complete the cells which contain diagonals, start again in the upper left-hand corner with 16 (since last cell counted in step one was 16) and count backwards across the first row, then across the second row, etc., as shown in Figure 2.

Magic Square IV

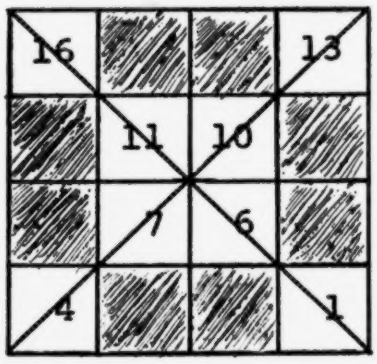


Figure 2

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figure 3

The completed square will look like Figure 3.

By omitting several of the numerals in the magic square, the students may be motivated to discover the pattern at the same time that they are reviewing the addition and subtraction facts.

Magic squares of 25 cells, 49 cells, or any odd number of cells may be made by following the same directions given for the square of 9 cells. Even-numbered magic squares beyond 16 are possible to make but because of the difficulty of making them, such squares are not included herein.

Further variations of magic squares can be found later in this text.

Mathematical Capers

1) Play this short and easy game called Ten.

Object of the game: The first person to arrive at the number 10 is the winner.

How to begin: Tell the students that you will always get to 10 before any member of the class.

Rules: Number of players: 2

Player A begins counting by saying 1 or (1, 2). (No player can say any more than two consecutive numbers.) Player B responds by saying the next one or two consecutive numbers.

Example

Player A	Player B
says → 1	says → 2,3
4	5
6,7	8,9
10	

How to win: The magic numbers for winning are 1, 4, or 7. The class should enjoy trying to discover the reasons why Player A can always be the winner. A variation of this game will be found with Natural Numbers.

2) Find the number of brothers and sisters a student has. (We assume here that no pupil has more than 10 brothers and no more than 10 sisters.)

1. Write down the number of brothers you have.
2. Double it and add one.
3. Multiply this by five.
4. Add the number of sisters you have.
5. Tell me your result.

When you hear the result, subtract 5 from it and you will have the number of brothers and sisters.

Example: His result is 17; subtract 5 and you get 12; the tens digit is brothers and the units digit is sisters (1 brother and 2 sisters).

Explanation of why this works:

Let x = number of brothers

y = number of sisters

1. x
2. $2x$
3. $2x + 1$
4. $5(2x + 1)$ or $10x + 5$
5. $10x + 5 + y$
6. Solution:

$$10x + 5 + y - 5$$

$$10x + y$$

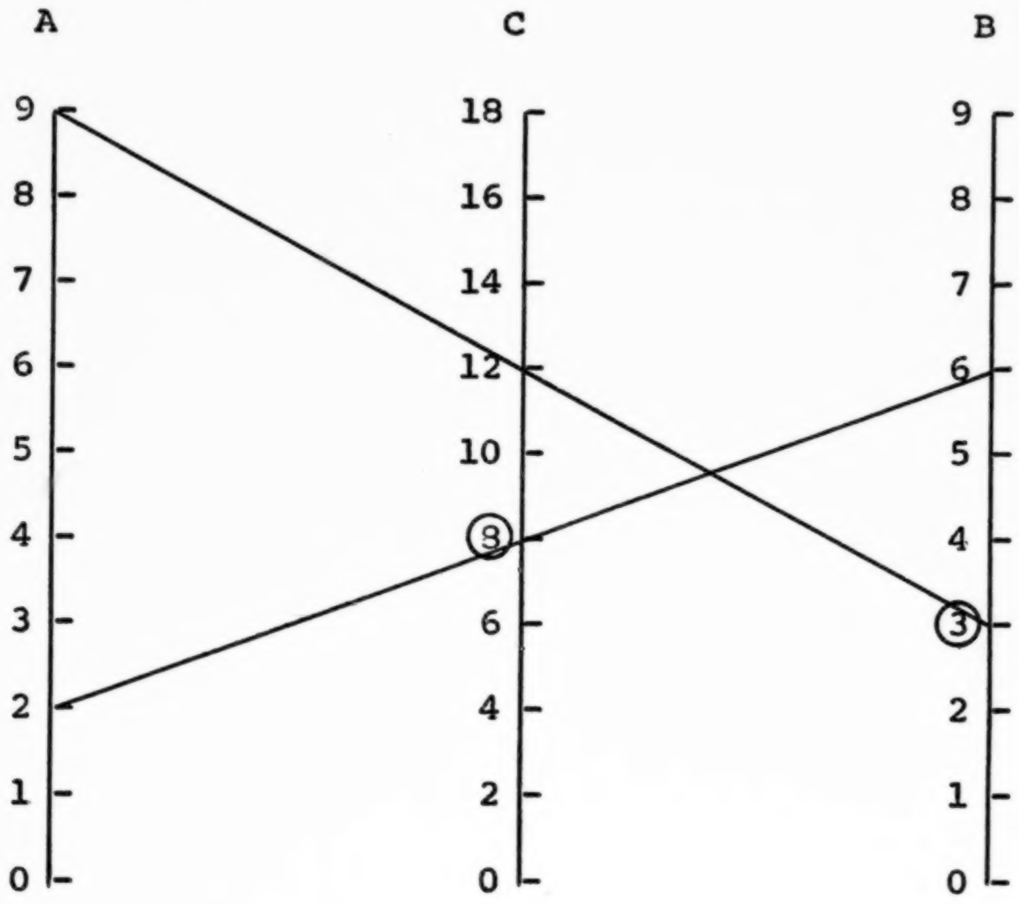
Nomographs Can Be Fun

This is a novel way to teach the operations of addition and subtraction.

Directions for making a nomograph for addition and subtraction:

1. Draw three vertical lines on a piece of regular graph paper with equidistant spacing.
2. Label the lines A, C, and B as shown in the diagram below.
3. Label scales A and B choosing a convenient scale.
4. Fill in scale C by adding scales A and B together at each level.

(Note: Scale C will be a doubled scale A or B.)

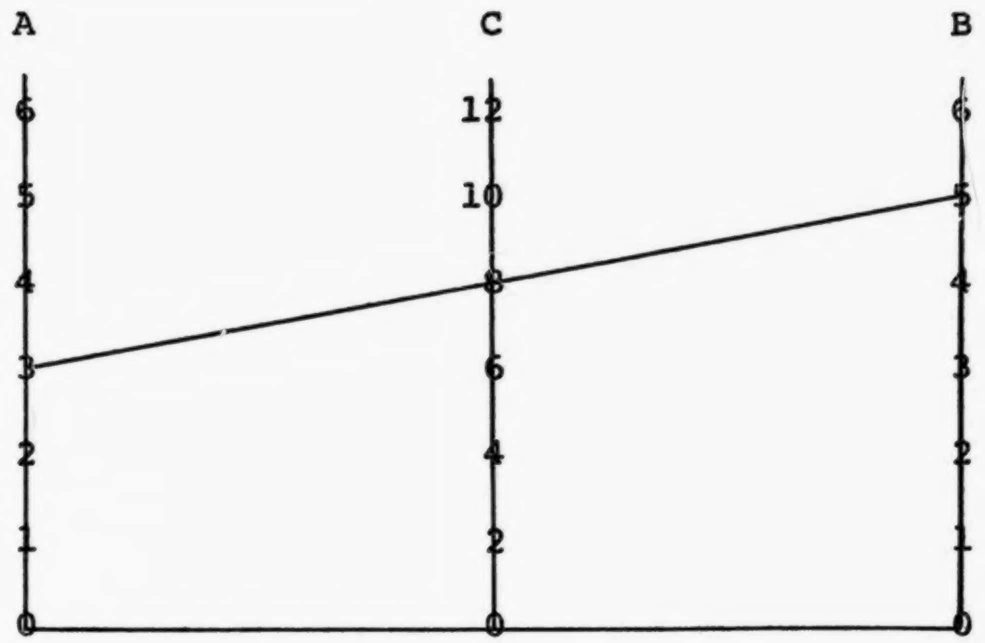


How to Use the Nomograph

To add 2 + 6 on a nomograph place a straight edge on the nomograph that passes through point 2 on scale A and point 6 on scale B. The straight edge will pass through the answer on scale C.

To subtract 9 from 12 on a nomograph place a straight edge on the nomograph that passes through point 12 on the C scale and point 9 on scale A. The straight edge will pass through the answer on scale B.

Why the Nomograph Works



Using the example 3 + 5 = 8 and looking at the figure that is formed, we see that it is a trapezoid. Since the nomograph is constructed with three vertical lines spaced equidistant apart, scale C is actually the median of this trapezoid. From the theorem: "The median of a trapezoid is equal to one-half the sum of the two bases," we have $C = \frac{(A + B)}{2}$. However, scale C is double scales A or B; therefore, the equation would become $C = \frac{(A + B)}{2} \times 2$ or, when simplified, $C = A + B$.

NOTE: for a class which has little success with the fundamental skills, this device could serve as a crude "slide rule": which pupils might be permitted to use on occasion. It could provide opportunities for success despite severe skill handicaps and could also permit focusing on pupil strengths despite the weaknesses which usually camouflage them.

Discovering Patterns

Searching for patterns can make a mathematics class interesting for the teacher and the student.

A. Find the patterns:

Answers for A (1 and 2)

1.

1	2	3	4	7	10	N	
7	10	13			34		16, 25, 3N+4

2.

4	5		10	14	20	N	6
17	26	37					101, 197, 401, N^2+1

3. Complete these processes:

1 x 8 + 1 = 9

12 x 8 + 2 = 98

123 x 8 + 3 = 987

1234 x 8 + 4 = 9876

12345 x 8 + 5 =

123456 x 8 + 6 =

1234567 x 8 + 7 =

12345678 x 8 + 8 =

123456789 x 8 + 9 =

B. Free style pattern searching:

1. Rule:

								Answers for 1
x+y	9	13				3		12, 10, 7, 17
x	6	9	8	8			10	5, $\frac{3}{2}$
y	3			2	2		7	4, 4, $\frac{3}{2}$
x-y	3		4		3	0		5, 6, 3

2. In this exercise the student must discover the rule in order to complete the blanks. (Hint: this has a procedure similar to that of 1.)

Rule: Answers for 2

x	9			2	$\frac{3}{4}$
y	3		3	6	
	27	56	63		
	6				
	3	$\frac{8}{7}$			$\frac{3}{2}$
	12				

x, 8, 21

y, 7, $\frac{1}{2}$

xy, 12, $\frac{3}{8}$

$(x-y)$, 1, 18, -4, $\frac{1}{4}$

$(x \div y)$, 7, $\frac{1}{3}$

$(x+y)$, 15, 24, 8, $\frac{5}{4}$

CHAPTER I

POINT OF VIEW

During the past ten years a great amount of attention has been given to writing mathematics curriculum materials for college-bound pupils. At the same time, relatively little has been said about the non college-bound pupil in the mathematics class. The present picture is made somewhat brighter for these pupils because the amount of attention being accorded them has been increasing. It can accurately be said that the teacher of academic mathematics has many sources from which ideas and information can be drawn, but the teacher of general mathematics has few sources of suggestions. It is hoped that this handbook will serve as one step on the path toward meeting this need.

The reader should be aware of what this handbook is and is not attempting to do. It is _not_ a mathematics text or course guide in any sense of these terms. Those who will be using the handbook will be following a variety of guides. It is _not_ a handbook for teaching mathematics to "slow learners," "underachievers," or "remedial classes," although it has something to offer in each of these cases. The goal was to prepare a booklet of ideas and approaches to mathematics for all teachers of general mathematics. It is known that characteristics of pupils in these classes will vary widely. Some of them will have better than average ability; some will have far less than average ability. Some will be motivated; others will lack all signs of motivation. We have endeavored to provide ideas which may be useful, in some form, with any of the very different pupils who make up general mathematics classes. Some of the material in the form in which it appears in this handbook is more appropriate for the slow pupil than for the average or above average. Other parts may be more suitable for the average pupil. Most of the ideas can be adapted with very little modification to either group of pupils.

It is a fundamental premise of this handbook that the general mathematics student, particularly the slow learner, is not so drastically different from other mathematics students as one might be led to believe by some things which have been written in recent years. General mathematics texts in the past have often contained very little mathematics. They have dwelt primarily on arithmetic and its social uses. There is often a tendency to associate general mathematics classes with problems related [illegible] discipline, learning difficulties, lack of motivation, etc. While these problems may occur more often in general mathematics classes, they can occur in any class given the proper circumstances.

Perhaps there is truth in the statement that able pupils can learn despite poor teaching techniques, but less able pupils require the most competent instruction. We need to get the finest possible teachers in all mathematics classes, general or academic. An adequate knowledge of the subject matter and creative teaching are as vital in general mathematics as in academic classes. The tendency to assign poorly prepared or inexperienced teachers to general mathematics classes probably contributes much to the creation of the problems which so many have come to accept as typical of general mathematics classes.

A laudable trend is one that requires all the mathematics teachers to share the load of general mathematics classes. Each teacher should be willing to accept the challenge to be at his very best as a teacher for one or two class periods per day. Class size should be kept small in general mathematics classes, even at the expense of larger class sizes in academic sections if necessary. General mathematics classes should not exceed 25 pupils with 20 as a more appropriate figure if possible.

Although the point of view has been expressed that general mathematics classes are not drastically different from other mathematics classes, it is probably appropriate to call attention to the fact that lower ability pupils tend to have shorter attention spans than do their more able peers. Therefore, teachers should provide greater variety and novelty in the lessons planned for these pupils. Although students in all mathematics classes need considerable practice in the skills they are trying to master, teachers should not rush blindly into long, dull drill sessions which completely ignore the factor of short attention spans. Drill can be made stimulating through thoughtful planning. It is hoped that this handbook will encourage and suggest approaches to this thoughtful planning. A serious effort has been made to provide a number of unique and interesting ideas which may motivate the learner and provide a great deal of useful practice. It is not expected that a class would completely subsist on a steady diet of recreational mathematics, but much good can be made of it.

Another suggestion that merits attention is that general mathematics classes should have daily lessons which are, if possible, relatively independent of each other. Such variety is particularly important for the less able pupils. These pupils have a shorter attention span, lower retention, and higher absence rates than do the able pupils. Carefully constructed, independent lessons can help minimize the adverse effects of these factors.

Finally, it is always important for students to have a successful experience with mathematics. Many general mathematics students have already failed in mathematics several times. The teacher should provide a chance for these students to succeed. Material such as that which is found in this handbook can be helpful in providing these opportunities.

It is hoped that all who read this handbook will find something interesting and useful in it. It is also hoped that each teacher will constantly turn to the many other sources of ideas and information which are now becoming available in the area of mathematics for the non college-bound pupil. Many adults today are inadequately trained in mathematics and they readily admit a genuine dislike for the subject. Given the best opportunity for success in mathematics, the next generation should have a better understanding of it and, even more important, a better attitude toward it.

CHAPTER II

NATURAL NUMBERS

The purpose of this section is to establish an understanding of the term "natural number." The term itself may or may not be used; if the student does not choose to use "natural" number, then use "counting" number. Do not use "whole" number.

Concrete items such as desks, chairs, and students should be counted by the students. This activity may suggest the name "counting" number. This is a good opportunity to introduce set notation if applicable to your class.

A. Use of natural numbers in a number game.

1) Name of game "First to 100 Wins."

2) Rules:

   a) Two players only.

   b) First player mentions a natural number not greater than 10.

      Second player mentions a natural number not greater than 10 and adds this to first player's number.

   c) This goes on until 100 is reached by one of the players.

A method that will guarantee your winning every time is based on the "key" numbers: 12, 23, 34, 45, 56, 67, 78, and 89. If the player chooses 4, the second should add 8 which places him on "key." You can probably see that when the second player reaches "key" number 89 there is nothing the first player can do about reaching 100.

4) If you wish to vary the game, this mathematical formula will give you the key numbers: x-(n+1). The x represents the winning number and n the number that can be added each time. In the game just played x = 100 and n = 10. Therefore, the keys are 100-11 = 89, 89-11 = 78, 78-11 = 67, 67-11 = 56, 56-11 = 45, 45-11 = 34, 34-11 = 23, 23-11 = 12.

Player A	Player B
says: 1	says: 8
12	22
23	32
34	41
45	55
56	57
67	76
78	80
89	93
100	

B. Computational aids for working with natural numbers.

1) Baseball Game.

a) Game of baseball to reinforce the basic number facts for addition, subtraction, multiplication, and division of natural numbers.

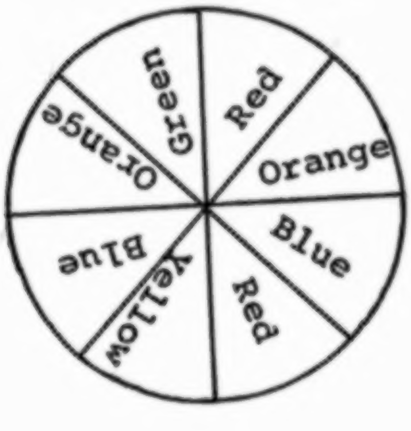
b) Equipment needed: poster paper to make flash cards and a color wheel; chalkboard on which to draw a ball field and scoreboard.

(1) Flash cards would consist of the basic addition and multiplication facts with the inverse operations included.

Set I	Set II	Set III	Set IV
Single	Double	Triple	Home Run
4	6	5	6
+3	x3	+□	x △
		7	42

(These are just examples to give the idea of increasing difficulty.)

(2) Color Wheel:



Red	Single
Blue	Double
Green	Triple
Yellow	Home Run
Orange	Out

(3) Scoreboard:

Inning	1	2	3	4	5	6	7	8	9
Team 1									
Team 2									

c) After choosing three field officials, divide the rest of the class into two teams.

Duties of officials:

(1) Determine by spinning the color wheel the type of hit the batter will attempt to make (an out is automatic when spun).

(2) Pick the appropriate flash card for the batter.

(3) Keep tally of runs and position of men on base.

As the year progresses the flash cards become more difficult: fractions, properties, percent equivalents, decimal percent equivalents, etc.

d) Added Feature:

If you play baseball and want to hit a home run, which is more important, a heavier bat or hitting the ball harder?

Hitting harder is more important than a heavier bat in trying to hit a home run. When you hit with something twice as heavy, it does twice as much good. But when you hit twice as hard, it does 4 times as much good.

Physics formula:

$$E = \frac{MV^2}{2}$$

Energy (E) equals half the product of the mass (M) times the square of the velocity (V).

If M is increased 3 times (M) = 3

If V is increased 3 times (V^2) = 9

2) Nomograph.

The nomograph, which appears in the Introduction, is an excellent device for reviewing addition and subtraction of natural numbers. (See page 13.)

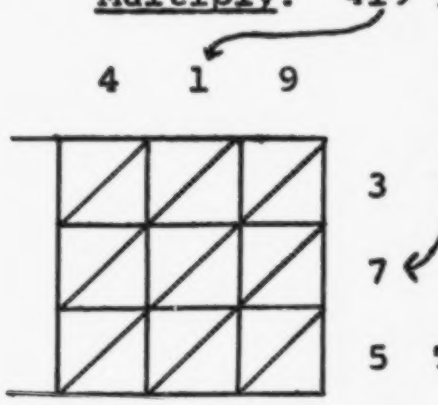
3) Magic squares are excellent for reviewing addition and subtraction of natural numbers. (See page 6.)

4) Suggestions for reviewing Multiplication of Natural Numbers.

   a) Lattice Method of Multiplication:

      (1) The use of the lattice method or form is simple in its operation because it presumes only a knowledge of the primary multiplication facts and the ability to add (no renaming). This was an early method used for multiplication.

Multiply: 419 x 375 = __________



419 3 digits

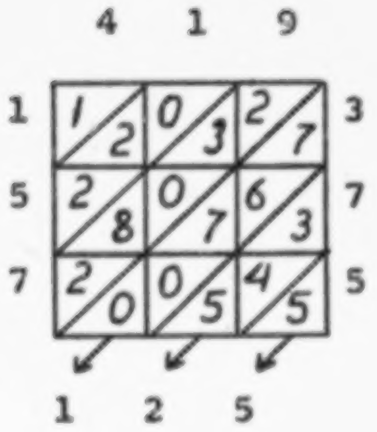
375 3 digits

Therefore: 3 x 3 or 9 squares

(2) Obtain products of the separate factors and write in the corresponding squares the products with the numeral representing the ones digit of the product below the diagonal and the tens digit of the product above the diagonal.

The method works because of distributive property.

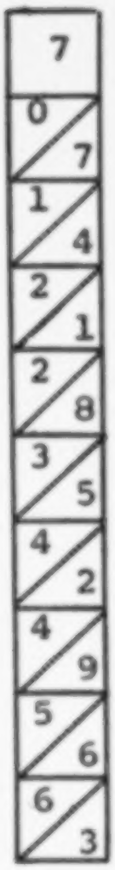
419
375
2700 = 300 x 9
3000 = 300 x 10
120,000 = 300 x 400
. . . .
630 = 70 x 9
700 = 70 x 10
28,000 = 70 x 400
. . . .
45 = 5 x 9
50 = 5 x 10
2000 = 5 x 400
. . . .
157,120



419 x 375 = 157,125

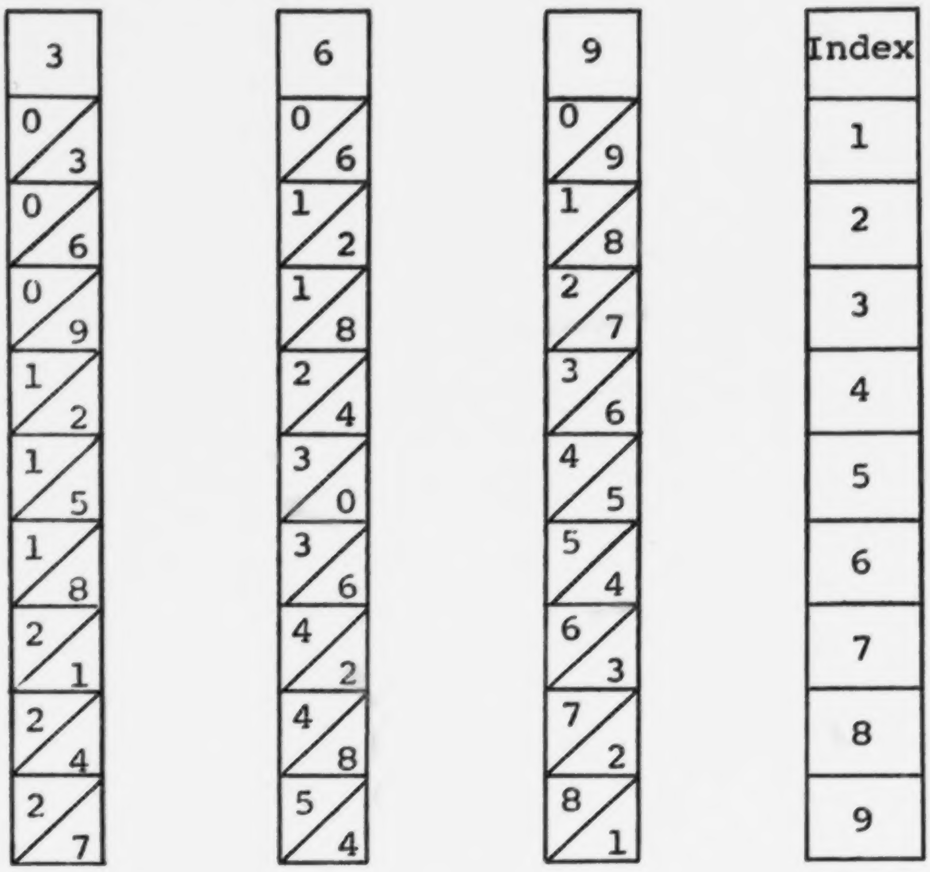
(3) The final product is obtained by adding the digits in the diagonals beginning with the diagonal in the lower right-hand corner and adding the numerals in each diagonal until all the additions have been performed. Numerals are carried into the next diagonal row of numerals where necessary as in ordinary addition. When the numerals thus obtained are read from left to right around the lattice frame, the product of the two given factors is obtained.

b) Napier's Bones:

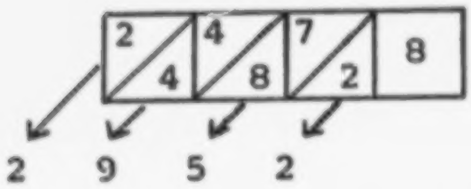


John Napier developed a method of multiplication closely related to lattice multiplication that is considered to be the forerunner of modern computing machines. Prepare a set of strips (bones) with multiples of the digits 1 through 9, together with an index stick bearing each of the numerals 1 to 9. The illustration shows the "bone" for 7.

To find the product of 8 x 369, place the "bones" headed 3, 6, 9, alongside the Index.


3
0 3
0 6
0 9
1 2
1 5
1 8
2 1
2 4
2 7
6
0 6
1 2
1 8
2 4
3 0
3 6
4 2
4 8
5 4
9
0 9
1 8
2 7
3 6
4 5
5 4
6 3
7 2
8 1
Index
1
2
3
4
5
6
7
8
9

To find the product we read along the diagonals in the row alongside 8 on the Index.


2 4
4 8
7 2
8
2
9
5
2

Therefore: 8 x 369 = 2952

The following computation shows how the product 427 x 369 is taken from the "bones." (400 x 369 + 20 x 369 + 7 x 369):

$$\begin{array}{r} 1476 \\ 0738 \\ \underline{2583} \\ 157563 \end{array} \qquad \text{or} \qquad \begin{array}{r} 2583 \\ 738 \\ \underline{1476} \\ 157563 \end{array}$$

c) The different patterns in the multiplication table are fun to discover. See how many the students can find and explain.

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

A few of the patterns that can be found:

(1) Increasing and decreasing patterns.

(2) Squaring functions

(3) Squaring of the binomial pattern

(4) Patterns on the diagonal

(5) Fractional equivalents when working with rational numbers.

C. Properties of natural numbers.

1) It is suggested that the teachers introduce the distributive property on the board or overhead projector by putting up examples one at a time and asking for individual responses (the name need not be used):

a) $(3 \times 4) + (3 \times 5) \longrightarrow 12 + 15 = 27$

b) $(6 \times 7) + (6 \times 3) \longrightarrow 42 + 18 = 60$

c) $(12 \times 2) + (12 \times 7) \longrightarrow$

d) $(11 \times 3) + (11 \times 6) \longrightarrow$

e) $(9 \times 7) + (9 \times 4) \longrightarrow$

After several of these exercises are completed with the class, the students may see the pattern and be ready with an answer as soon as the problem is written. At this point pupils are ready to discover the distributive property by using this type of problem and comparing the answers to the first set of exercises.

$3 \times (4 + 5) \longrightarrow 3 \times 9 = 27$

$6 \times (7 + 3) \longrightarrow 6 \times 10 = 60$

$12 \times (2 + 7) \longrightarrow 12 \times 9 = 108$

From comparing the answers, a short cut for solving this type of problem may be discovered. Use this discovery to solve such problems as:

$(11 \times 4) + (11 \times 23) \longrightarrow 11\ (4 + 23) = 297$.

Now check your result by actual computation.

$44 + 253 = 297$

Now have the short-cut method explained. Explanation given by students is usually rough but can be polished by the rest of the class with teacher guidance. You want the students to realize that the distributive property can be used in two ways:

First: $a(b + c) = ab + ac$

Second: $ab + ac = a(b + c)$.

The second method is seldom found written in the system of natural numbers but is a very important concept in algebraic factoring.

2) Suggested type of homework problems for the distributive property:

a) Show that $3(5 + 4) = (3 \times 5) + (3 \times 4)$.

b) Using the distributive property, rename the following sum:

$20 + 25 = (5 \times 4) + (5 \times 5) = 5(4 + 5)$

c) Using the distributive property, find the following product:

(1) $6 \times 53 = 6(50 + 3) = 300 + 18 = 318$

(2) $23 \times 25 = (20 + 3)\ (20 + 5) = 20(20 + 5) + 3(20+5) = 400 + 100 + 60 + 15 = 585$

3) Culminating activity:

5 x 24,736

Have students discover that the distributive property holds for not only a(b + c) but also for a(b + c + d + . . .).

4) Extension of Multiplication Table by using Distributive Property. To show: a(b + c) = ab + ac

a) 1 (7) = 7
b) 2 (7) = 14
c) 3 (7) = 21
d) 4 (7) = 28
e) 5 (7) = 35
f) 6 (7) = 42
g) 7 (7) = 49
h) 8 (7) = 56
i) 9 (7) = 63
j) 10 (7) = 70
k) 11 (7) = 77
l) 12 (7) = 84

(1) Can we use the distributive property and the above list to get a value for 17 (7)?

(2) List some combinations that would give 17 (7):

11(7) + 6(7) = 17(7)
12(7) + 5(7) = 17(7)
9(7) + 8(7) = 17(7)
10(7) + 7(7) = 17(7)

(3) Let's take the combination

12(7) + 5(7) = 17(7)

What is the product of 12(7) as indicated on the list? (84)

What is the product of 5(7) as indicated on the list? (35)

Will the sum of 84 and 35 equal 17(7)? (Yes, but have students multiply to check answer.)

(4) Look at 9(7) + 8(7) and add their products. What is your sum? (119) What do you observe? (same answer)

(5) Do you think we would get this result for any grouping of 17 such as 4(7) + 6(7) + 7(7)? (Add products 28 + 42 + 49 and get sum of 119.)

(6) The sum of the first five products in this table will equal how many sevens? (Answer: 15(7).)

(7) The sum of which products in this table would be 37(7)? (various results)

(8) Is multiplication distributive over addition? Give an example.

(9) Is addition distributive over multiplication? Does 8 + (7 x 5) = (8 + 7) x (8 + 5)?

D. Discover the pattern.

Rules to be discovered:

1)

a	4	8	0	10	5	11	16	9	2a = b
b	8	16	0	20					

2)

a	2	7	0	9	6	18	16	11	a + 8 = b
b	10	15	8	17					

3)

a	3	7	0	10	5	8	6	1	2a + 12 = b
b	18	26	12						

4)

	3	14	2			6	12	15	a − b
a	6	15		7		11			
b	3	1	8	2	5		4		
	9	16	18		12			25	a + b

5)

	3	4		3			29	18	a − b
	9	8	8		11	12	31		a + b
a	6	6			9				
b	3	2	4	3		5			
	18	12		18					ab
	25	19			25	92		47	ab + 7

6)

a	3	7	0	10	5	8	6	1	a + 2 = b
b	5	9	2						

7)

a	3	7	0	10	5	8	6	1	$2a + 4 = b$
b	10	18	4						

8)

a	4	2	0	10	9	5	6	1	$a^2 + 1 = b$
b	17	5	1	101					

9)

	6	12			2			24	$2(a - b)$
	3	6	7			11	25		$a - b$
a	5	7	10	20	8	16	25		
b	2	1	3	10				2	

10)

	16	20	12	16			24		$2(a + b)$
	15	24				36			ab
	2	2	2	6	0			21	$b - a$
a	3	4	2	1					
b	5	6	4						
	8	10		8	2	12			$a + b$
	18	27	11					103	$ab + 3$
	6	6	6	18			18		$3(b - a)$

CHAPTER III

INTEGERS

A. Motivational Experiences for Extending the Number Line.

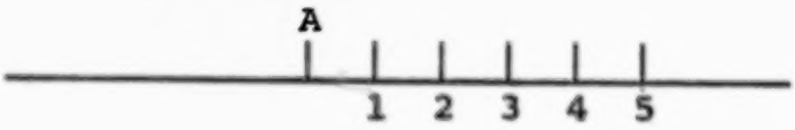
1. If students understand the use of the Centigrade and/or Fahrenheit thermometer, use one to illustrate the rising and falling of temperature. Be sure to emphasize numbers below zero. Model thermometers can be purchased or made.
2. Discuss going up and down on an elevator from different floor levels. Ask such questions as, "If you start on an elevator at floor A, go up five floors and come down six floors, where are you with reference to your starting point?" (Answer: one floor below A.) Use several examples of this type to emphasize the need for numbers other than natural numbers. Be sure to include answers that would give zero, answers above and answers below the starting floor. Ask: "How can we represent numbers above the starting point?" "How can we represent numbers below the starting point?" "Could all this be represented by using a number line?"
3. In a football game the ball is at the line of

scrimmage. On the first play there is a gain of seven yards; on the second play, a loss of nine yards. Where is the ball in reference to the original line of scrimmage? (Use several examples of this type.)

4. In reference to a rocket problem, how do we designate blast off time? (T or zero) How do we designate 7 seconds before blast off time? (T - 7) How do we designate 7 seconds after blast off time? (T + 7) Could we represent this information on a number line?

B. Concept of Integers.

1. Recall the number line with natural numbers.



a. A is another point as far to the left of 1 as 2 is to the right of 1. What should we call this point? (Answer: Zero or origin.) Emphasize that 0 is the origin or starting point.

b. How do you think you could locate the point to the left of zero which is the same distance as 1 is to the right of zero? Can you do the same for a point 2 units away? 3 units? etc.? Do you think you could always find a point the same distance to the left of zero as a natural number is to the right of zero? NOTE: These "opposite" numbers may be called "additive inverses" in the secondary school.

c. What names can we call these points? (Answer: -1, -2, -3, etc.) (If they choose letters, show that letters are a finite set and we need an infinite set for naming these points.)

2. In working with the number line it is important to stress that movement to the right is in a positive direction, and movement to the left is in a negative direction. There are other ways of expressing these directions.

Suggested terminology:

Positive	Negative
Forward	Backward
Up	Down
Above	Below
Gain	Loss
Right	Left
East	West
North	South

C. Computations on the Number Line.

1. Addition:

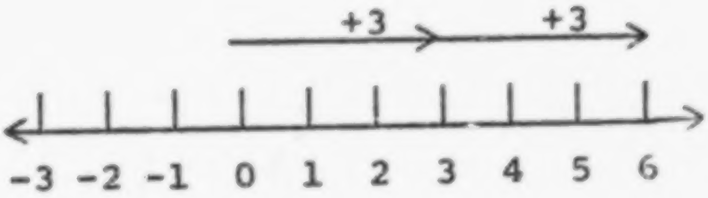
An integer may be represented as an arrow. Point out the specific meaning of the arrow in this case as opposed to the definition of a ray if the students are familiar with a ray. The length of the arrow is the "distance" the integer is from the origin.

To add two integers draw the arrow for the first integer. Remember that the starting point is the origin. Then draw the second arrow with its tail at the tip of the first arrow. The tip of the second arrow is on (over) the integer which is the sum of the two integers.

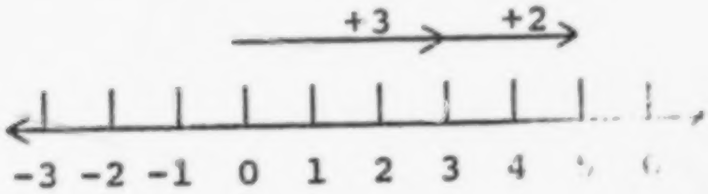
If you do not want to use arrows, you may count on the number line.

a. Examples of addition: Zero (origin) is always the starting point.

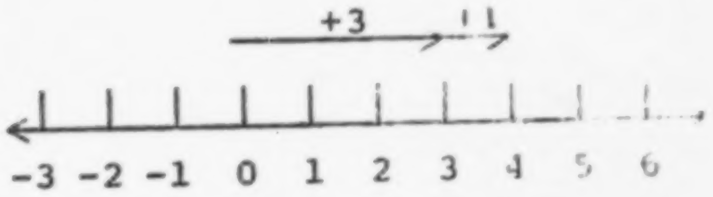
1) 3 + 3 = 6


+3
+3
-3 -2 -1 0 1 2 3 4 5 6

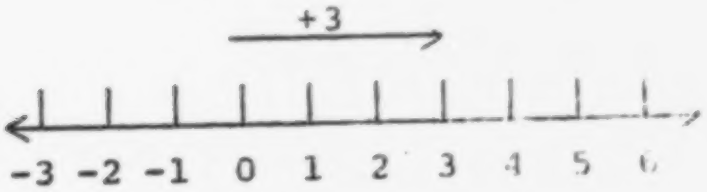
2) 3 + 2 = 5


+3
+2
-3 -2 -1 0 1 2 3 4 5 6

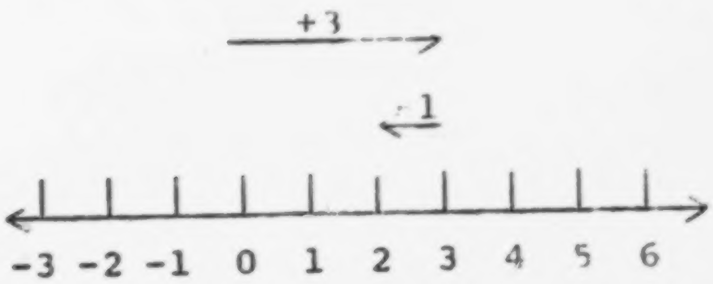
3) 3 + 1 = 4


+3
+1
-3 -2 -1 0 1 2 3 4 5 6

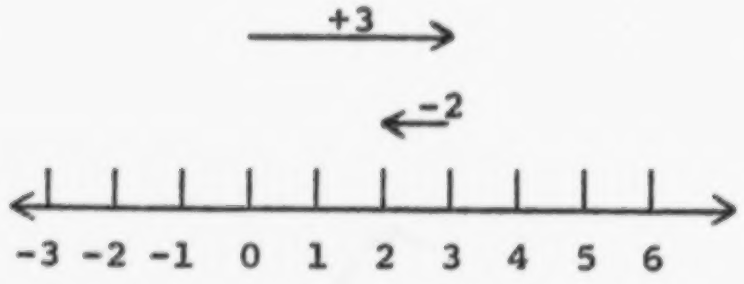
4) 3 + 0 = 3


+3
-3 -2 -1 0 1 2 3 4 5 6

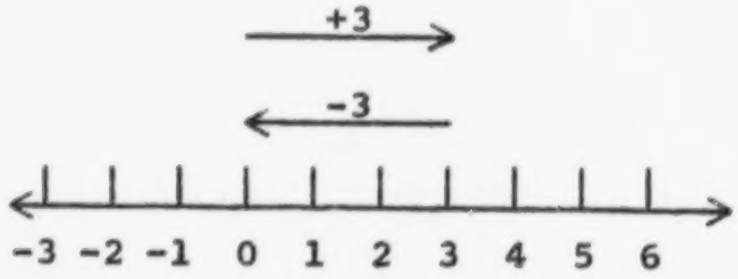
5) 3 + (-1) = 2


+3
-1
-3 -2 -1 0 1 2 3 4 5 6

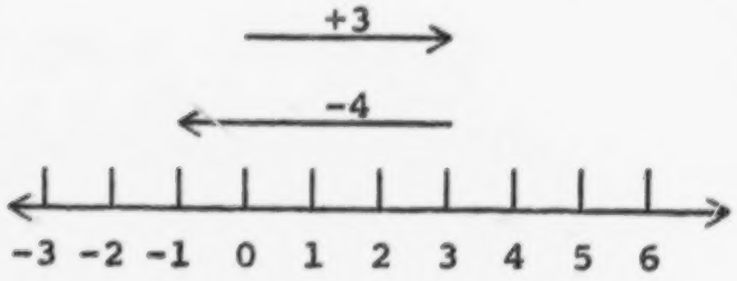
6) 3 + (-2) = 1


+3
-2
-3 -2 -1 0 1 2 3 4 5 6

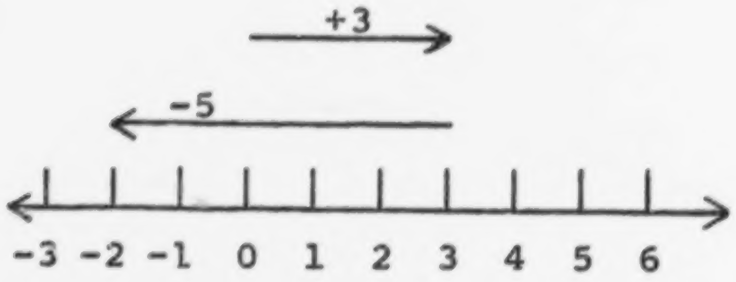
7) 3 + (-3) = 0


+3
-3
-3 -2 -1 0 1 2 3 4 5 6

8) 3 + (-4) = -1

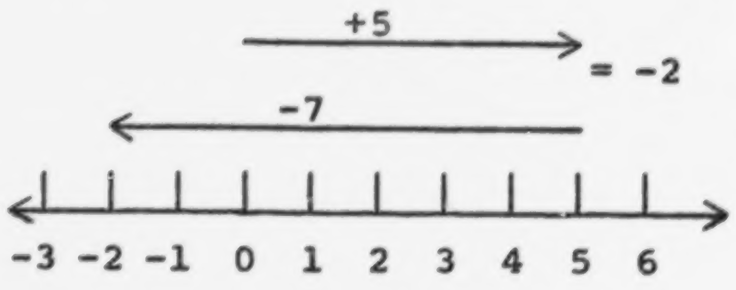

+3
-4
-3 -2 -1 0 1 2 3 4 5 6

9) 3 + (-5) = -2

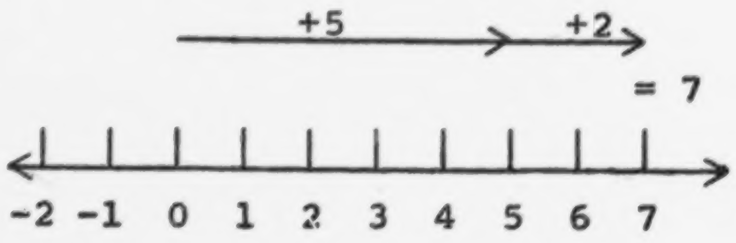

+3
-5
-3 -2 -1 0 1 2 3 4 5 6

b. Other addition examples:

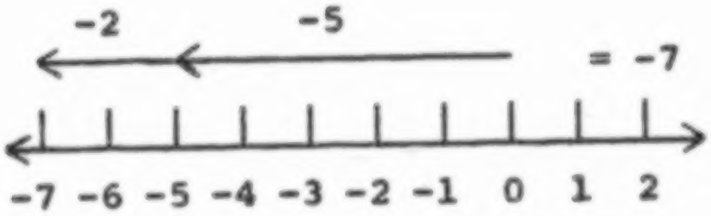
1) Add +5 and -7


+5
= -2
-7
-3 -2 -1 0 1 2 3 4 5 6

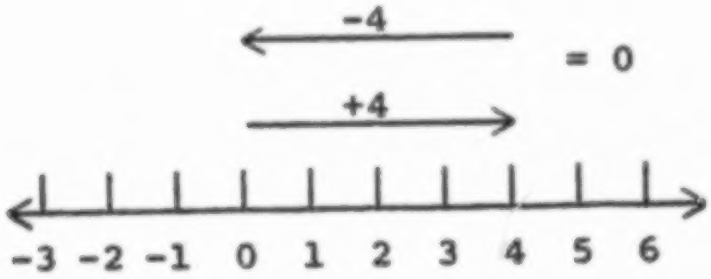
2) Add +5 and +2


+5
+2
= 7
-2 -1 0 1 2 3 4 5 6 7

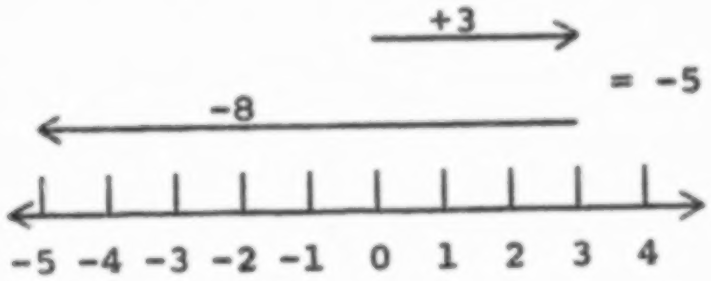
3) Add -5 and -2


-2
-5
= -7
-7 -6 -5 -4 -3 -2 -1 0 1 2

4) Add -4 and +4


-4
= 0
+4
-3 -2 -1 0 1 2 3 4 5 6

5) Add +3 and -8


+3
= -5
-8
-5 -4 -3 -2 -1 0 1 2 3 4

Suggested exercises using the number line.

Locate	Move	Stop at
+3	+5	?
+3	?	+8
+5	?	+10
+5	-2	?
+5	?	+3
+7	?	+3
+11	?	+6
+4	?	-4
-5	?	-10
-5	?	+5
-10	?	+3
-8	?	0

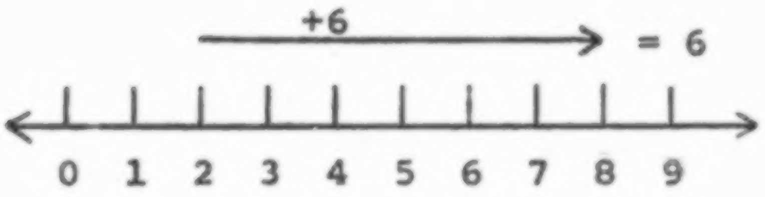
Let students discover the rules for adding numbers with like or unlike signs by use of activities such as previously mentioned.

2. Subtraction:

   a. Definition:

      a - b = N means b + N = a

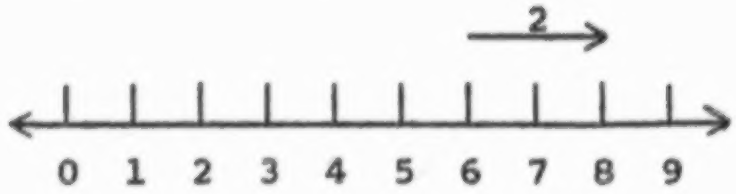
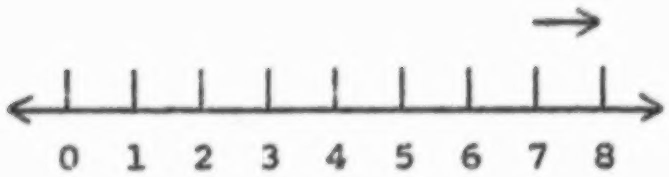
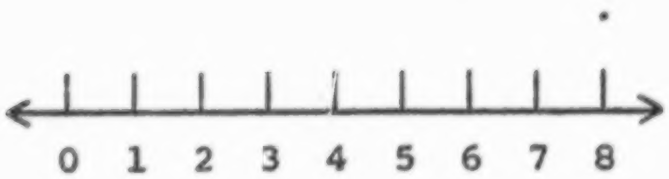
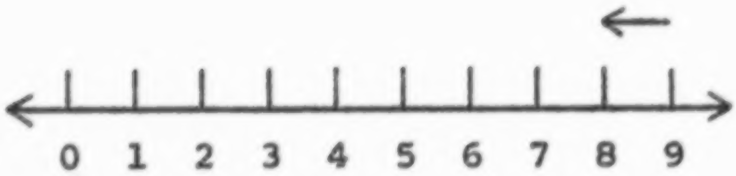
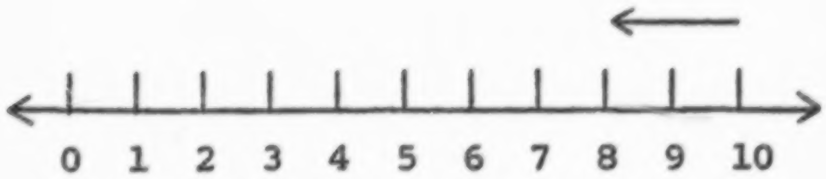
      8 - 2 = N means 2 + N = 8



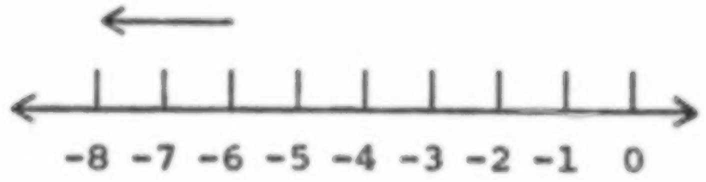
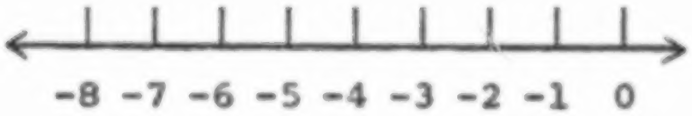
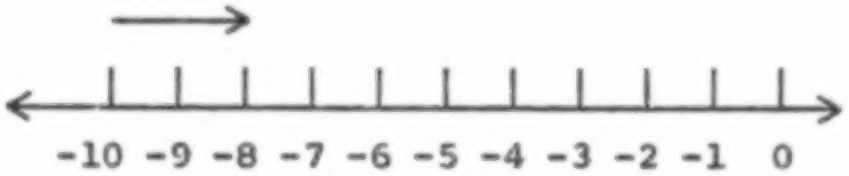
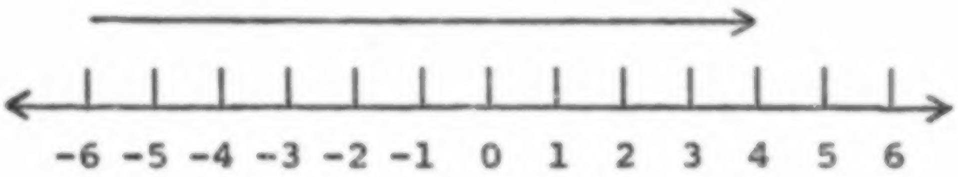
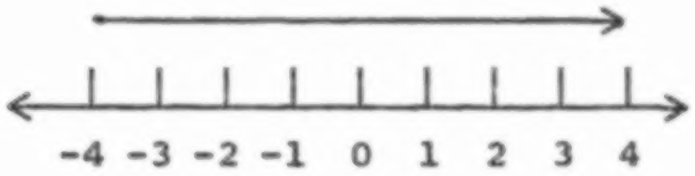
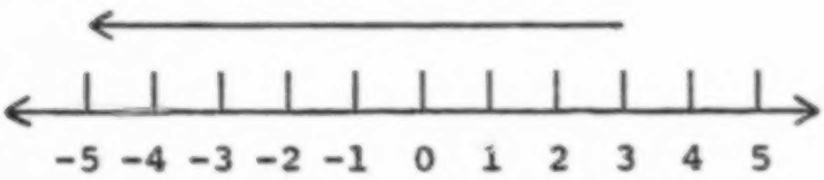
NOTE: Emphasize that the starting point on the number line for subtraction is the subtrahend and the end point is the minuend. The direction will give the sign of the answer and the distance between the numbers will give the answer. Students may be unfamiliar with the terms subtrahend and minuend; if so, use terms familiar to your class.

   b. Let the class discover that the definition of subtraction of natural numbers holds for the system of integers by use of the following:

1) Subtraction patterns. Use of number line.

Problem	Meaning	Number Line	Answer
a) (+8) - (+6) = N	6 + N = 8	 2 0 1 2 3 4 5 6 7 8 9	= 2
b) (+8) - (+7) = N	7 + N = 8		= 1
c) (+8) - (+8) = N	8 + N = 8		= 0
d) (+8) - (+9) - N	9 + N = 8	 0 1 2 3 4 5 6 7 8 9	= -1
e) (+8) - (+10) = N	10 + N = 8	 0 1 2 3 4 5 6 7 8 9 10	= -2

2) Other subtraction examples.

Problem	Meaning	Number Line	Answer
a) (-8) - (-6) = N	-6 + N = -8		= -2
b) (-8) - (-8) + N	-8 + N = -8		= 0
c) (-8) - (-10) = N	-10 + N = -8	 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0	= 2
d) (+4) - (-6) - N	-6 + N = 4	 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6	= 10
e) (+4) - (-4) = N	-4 + N = 4		= 8
f) (-5) - (+3) = N	3 + N - -5	 -5 -4 -3 -2 -1 0 1 2 3 4 5	= -8

c. NOTE: Whenever a smaller number is subtracted from a larger number, the result is always positive. Be sure students understand that we refer to numbers, not absolute values. If the larger number is subtracted from a smaller number, the result is always negative.

d. Suggested types of practice questions to discover rules of subtraction:

1) $(+3) - (-2) = +5$

   $(+3) + (+2) = +5$

   $(+3) - (-2) = (+3) + (+2)$

2) $(-6) - (-2) = -4$

   $(-6) + (+2) = -4$

   $(-6) - (-2) = (-6) + (+2)$

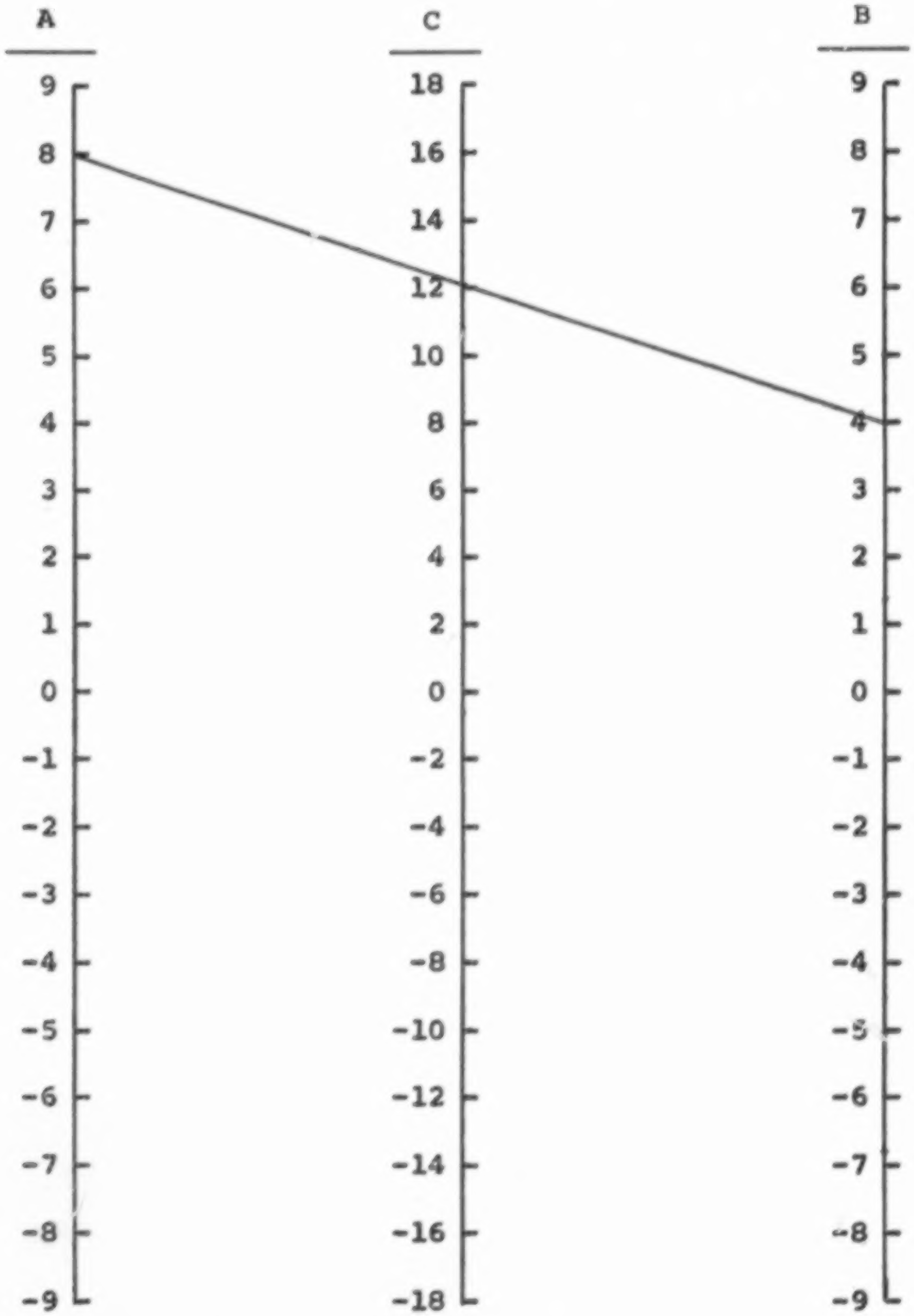
   Similar examples will reinforce the idea that addition and subtraction are inverse operations. Therefore, subtraction may be defined as "adding the additive inverse of the number to be subtracted."

e. Let the students discover the rules for subtracting numbers with like or unlike signs by use of activities previously mentioned.

3. By extending the Nomograph, the operations of addition and subtraction of integers may be included.

Nomograph for addition, subtraction of integers.

$A + B = C$ $C - B = A$ $C - A = B$


A
C
B
9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9
18 16 14 12 10 8 6 4 2 0 -2 -4 -6 -8 -10 -12 -14 -16 -18
9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9

Try these problems:

1) 8 + 4 = 12

2) 9 + -3 = 6

3) 0 + -8 = -8

4) 6 + -6 = 0

5) -8 + -4 = -12

4. Multiplication of integers:

   a. It is not wise to use a number line approach for multiplying two negative numbers.

   b. Discovery exercises for multiplication of integers:

      1) Review concept of multiplication table of natural numbers to establish the pattern.

x	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

      2) If your class has difficulty in grasping all concepts at the same time, add one number at a time to the multiplication chart of natural numbers. First add 0 to get multiples of whole numbers; then add -1, -2, etc., for integers.

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

x	-1	0	1	2	3
-1		0			
0	0	0	0	0	0
1		0	1	2	3
2		0	2	4	6
3		0	3	6	9

3) Multiplication Chart for Integers to help in discovering rules for the multiplication of numbers with like or unlike signs. (Let the students do the chart.)

	x	-3	-2	-1	0	1	2	3	
	-3	9	6	3	0	-3	-6	-9	
Section D	-2	6	4	2	0	-2	-4	-6	Section B
	-1	3	2	1	0	-1	-2	-3	
	0	0	0	0	0	0	0	0	
	1	-3	-2	-1	0	1	2	3	
Section C	2	-6	-4	-2	0	2	4	6	Section A
	3	-9	-6	-3	0	3	6	9	

Directions:

a) Zero times zero will give zero; therefore, the zero row and column can be filled in at once.

b) The right hand lower section is the

multiplication chart for natural numbers. Fill in Section A.

c) Observe a pattern to be followed by moving up each of these columns; each value in the last column decreases by 3; therefore, the column can be added to by using -3, -6, and -9. Discover similar patterns for the other two columns and complete Section B.

d) Discover similar patterns moving to the left in each row of Section A and complete filling in Section C.

e) Use either Sections B or C to discover a pattern that moves into Section D. In either case there is an increase in value. Complete Section D.

f) Discover the rules for multiplying signed numbers.

    (1) Section A shows that (+) (+) = +

    (2) Sections B and C show that (+) (-) = -

    (3) Section D shows that (-) (-) = +

g) Since multiplication and division are inverse operations, the rules discovered for multiplication will also hold true for division of integers.

5. This idea can be used to review the operations of addition and multiplication for any number system.

   a. Directions for making a "numerical isometric" for addition:

+	7	-12	6
-2			
5			
-7			

Figure 1

+	7	-12	6
-2	5	-14	4
5	12	-7	11
-7	0	-19	-1

Figure 2

+	7	-12	6	
-2	5	-14	4	-5
5	12	-7	11	16
-7	0	-19	-1	-20
	17	-40	14	Total -9

Figure 3

1) Draw a square with 9 cells.
2) Choose six numbers at random and assign one to each row and column as done for an addition table (Figure 1).
3) Complete the addition table (Figure 2).
4) Find the sum of each column and each row (Figure 3).

5) Discover that summation of the column totals will equal the summation of the row totals.

b. Directions for making a numerical isometric for multiplication are the same as for addition when the word multiplication is inserted for addition:

Example:

x	1	-2	3	
-1	-1	2	-3	6
2	2	-4	6	-48
-4	-4	8	-12	384
	8	-64	216	Total -110,592

Comment:

Only one example of this type should be used. Students could become bored if more are used because of complicated multiplication.

c. Directions for making a numerical isometric which involves a combination of multiplication and addition:

x	2	-3	3
4			
5			
-1			

Figure 1

x	2	-3	3
4	8	-12	12
5	10	-15	15
-1	-2	3	-3

Figure 2

x	2	-3	3	
4	8	-12	12	8
5	10	-15	15	10
-1	-2	3	-3	-2
	16	-24	24	16

Figure 3

1) Draw three squares as above.

2) Choose six numbers at random and assign one to

each row and column as done for a multiplication table (Figure 1).

3) Complete the multiplication table (Figure 2).

4) Find the sum of each column and each row (Figure 3).

5) Discover that summation of the column totals will equal the summation of the row totals.

Why it works:

The idea will work each time because the students are adding or multiplying the same nine numbers (body of table) in a different order and using different groupings. (Commutative and associative properties for addition or multiplication.)

CHAPTER IV

RATIONAL NUMBER SYSTEM

A. Ideas for Transition from Integers to Rational Numbers.

1. Now that we have established a number line showing the intervals represented by integers, let us look at the interval between 0 and 1. Zero corresponds to a point and one corresponds to a point. Do you think there are any points between 0 and 1 that have not been identified with a numeral? At this time in the development of the lesson, one of the "half-way paradoxes" could be used. Let the wall of the classroom represent 1 and the door 0. Ask a student to move from the wall to the door, stopping each time at a place which represents half the remaining distance. These stopping places illustrate that there are some points between 0 and 1. If the interval from 0 to 1 is one unit, what would you call the point midway between these two? Other points between these points may be identified. You may see that by following this procedure, an infinite number of points can be identified. What kinds of numbers correspond to these points? (Expected

answer: fractions.) These may be called rational numbers; you will observe that the top and bottom numerals are integers. They have the general form n/d, $d \neq 0$. If you replace n and d with different integers, you will get many rational number names such as:

$$\frac{4}{7}, \frac{3}{5}, \frac{-7}{11}, \frac{8}{16}, \frac{-5}{1}, \frac{6}{2} \ldots$$

Do you recognize these numbers? Can any be simplified? List the ones that can be simplified and give their simplest name.

2. A rapid check will let a teacher know if his class understands the meaning of a fraction. Emphasize that a fraction is not a part of a whole but a ratio of two integers. Thus, the rational number system is made up of any number that can be expressed as a fraction where the denominator is not zero.

B. Operational Aids.

1. Discover that in the addition of fractions the denominators are not added.
    a. Rename some whole numbers using fractional numerals, e.g., $6 = \frac{12}{2}$, $3 = \frac{15}{5}$, etc.
    b. Consider $1 + 1 = 2$.
        1) Rename each 1 with the same fractional numeral.

$$\frac{3}{3} + \frac{3}{3} = 2$$

2) Rename 2 as a fractional numeral having the same denominator as was used for each 1.

$\frac{3}{3} + \frac{3}{3} = \frac{6}{3}$

3) Compare the left member of the equation with the right member of the equation to notice that the numerator of the right member is the sum of the numerators of the left member, but the denominator is not the sum of the denominators of the left. Recall that we got $\frac{6}{3}$ by renaming the 2 rather than by adding $\frac{3}{3}$ and $\frac{3}{3}$.

*This is a good example of a problem in which the necessary result is known and we are observing to see the pattern of operation required to get that result.

Another example:

$2 + 3 = 5$ $\qquad 2 = \frac{4}{2}$, $3 = \frac{6}{2}$, $5 = \frac{10}{2}$

$\frac{4}{2} + \frac{6}{2} = \frac{10}{2}$

Do several examples of this type and then have students discover that in adding two fractions with like denominators, numerators are added and the sum placed over the common denominator.

Suggested activities:

$$\frac{3}{7} + \frac{4}{7} = \frac{7}{7} \qquad \frac{5}{12} + \frac{2}{12} = \frac{7}{12} \qquad \frac{1}{3} + \frac{2}{3} = \frac{3}{3}$$

$$\frac{2}{11} + \frac{3}{11} = \frac{5}{11} \qquad \frac{3}{17} + \frac{5}{17} = \frac{8}{17}$$

Here are some fractional magic squares:

$\frac{8}{9}$		$\frac{6}{9}$
$\frac{3}{9}$	$\frac{5}{9}$	$\frac{7}{9}$
	$\frac{9}{9}$	

$\frac{8}{11}$		
	$\frac{5}{11}$	$\frac{7}{11}$
	$\frac{9}{11}$	$\frac{2}{11}$

2. Fractional pattern development.

   a. $\frac{4}{10} + \frac{5}{10} = \frac{9}{10}$

   b. $\frac{2}{5} + \frac{5}{10} = \frac{9}{10}$

   c. $\frac{2}{5} + \frac{1}{2} = \frac{9}{10}$

      1) In step a. note that the method of adding fractions with like denominators is used.

      2) In step b. the first term has been renamed.

      3) In step c. the second term has been renamed.

      4) Observe in all 3 steps that the answer must remain the same.

Through discovery we can see that it is necessary that fractions be renamed with like denominators in order to add. A practice problem for addition of fractions can be given in this form:

Is this a magic square?

$\frac{17}{30}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{2}{5}$	$\frac{7}{15}$	$\frac{8}{15}$
$\frac{13}{30}$	$\frac{9}{15}$	$\frac{11}{30}$

NOTE: Subtraction of fractions can be treated as an addition problem.

$$\frac{3}{4} - \frac{1}{4} \rightarrow \frac{1}{4} + \frac{\square}{4} = \frac{3}{4}$$

$$\frac{1 + \square}{4} = \frac{3}{4}$$

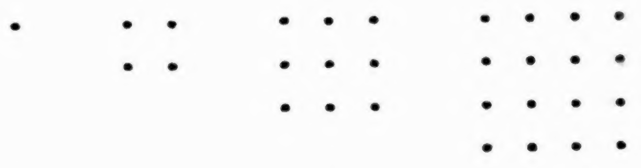
Therefore: $\square = 2$

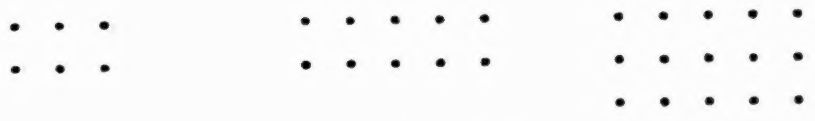
$$\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$$

3. Factoring to find lowest common multiple.

   a. Discovering prime numbers.

      The students may have seen these rectangular patterns before; if not, let them discover the patterns for themselves.





(A line is not considered a rectangular pattern.)

Are there any numbers that cannot be used to form rectangular patterns? (2, 3, 5, 7, 11, 13.) These numbers have no divisors except themselves and 1 and are called prime numbers.

b. Another interesting method of sorting out prime numbers is called the "Sieve of Eratosthenes."

1) List natural numbers in order of increasing magnitude beginning with one. (It is not necessary to list 100 numerals to show this but it is helpful to have an (n x 10) array.)

2) Cross out all even numbers except 2.

3) Eliminate all numbers divisible by 3, 5, and 7, except these numbers themselves.

4) The remaining 26 numbers are primes.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

c. Show that a composite number can usually be factored in more than one way. However, there is only one way to express it using prime numbers.

Example: 40 may be factored as 8 x 5, or 4 x 2 x 5, or 10 x 4 but in each of these products the factors are not all prime numbers. Each number must be factored that is not prime until the result 2 x 2 x 2 x 5 is reached.

d. To find the LCM:

Consider the fractions $\frac{1}{20}$, $\frac{1}{30}$, and $\frac{1}{40}$. Find the least common multiple for the denominators 20, 30, and 40.

$20 = 2 \times 2 \times 5$

$30 = 3 \times 2 \times 5$

$40 = 2 \times 2 \times 2 \times 5$

Method:

1) Write each denominator as a prime factorization.
2) Write a product using each prime factor the greatest number of times it occurs in any single denominator.

In this example the greatest number of 2's in any one denominator is three. The greatest number of 3's is one. There is one 5. The LCM is $2 \times 2 \times 2 \times 3 \times 5 = 120$. This LCM becomes the LCD when adding fractions with denominators of 20, 30, 40.

4. To add fractions having different denominators.

   a. Consider $\frac{1}{20} + \frac{1}{30} + \frac{1}{40}$

   We have already noted that the LCD is 120.

   $\frac{1}{20} \times (1) + \frac{1}{30} \times (1) + \frac{1}{40} \times (1)$ names the same sum since 1 is the identity for multiplication.

   $$1 = \frac{\text{LCD}}{\text{LCD}} = \frac{120}{120}$$

   $\frac{1}{20} \times \left(\frac{120}{120}\right) + \frac{1}{30} \times \left(\frac{120}{120}\right) + \frac{1}{40} \times \left(\frac{120}{120}\right)$ becomes

   $$\frac{6}{120} + \frac{4}{120} + \frac{3}{120} = \frac{13}{120}$$

   b. Could we rename 1 in a way other than $\frac{\text{LCD}}{\text{LCD}}$ and would there be a value in so doing? We know that 120 is the LCD.

   $\frac{1}{20} \times (1) = \frac{\quad}{120}$ $\qquad$ $\frac{1}{30} \times (1) = \frac{\quad}{120}$ $\qquad$ $\frac{1}{40} \times (1) = \frac{\quad}{12}$

   $\frac{1}{20} \times \left(\frac{6}{6}\right) = \frac{6}{120}$ $\qquad$ $\frac{1}{30} \times \left(\frac{4}{4}\right) = \frac{4}{120}$ $\qquad$ $\frac{1}{40} \times \left(\frac{3}{3}\right) = \frac{3}{120}$

   $$\frac{6}{120} + \frac{4}{120} + \frac{3}{120} = \frac{13}{120}$$

5. Division of fractions.

a. Reinforce process of multiplication of fractions:

$$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = \frac{ace}{bdf}$$

Special cases:

$\left(\frac{2}{3}\right) \times \left(\frac{3}{2}\right) = 1$ $\qquad$ $\frac{7}{3} \times (\ ) = 1$

$5 \times (\ ) = 1$ $\qquad$ $\frac{1}{a} \times (\ ) = 1$

$\frac{1}{7} \times (\ ) = 1$ $\qquad$ $2\frac{3}{5} \times (\ ) = 1$

$4\frac{1}{3} \times (\ ) = 1$

Observe these two factors that give us one each time. What is true about them? These fractions are called reciprocals or multiplicative inverses.

b. Suggested methods for teaching division of fractions:

1) $\frac{4}{5} \div \frac{2}{3}$

Rewrite as a complex fraction: $\dfrac{\frac{4}{5}}{\frac{2}{3}}$

Multiply by identity element of $1 \times \dfrac{\frac{4}{5}}{\frac{2}{3}}$

Use denominators to find LCM to use for identity element:

$$\frac{\frac{4}{5}}{\frac{2}{3}} \quad \frac{15}{15} = \frac{\frac{4}{5} \times \frac{15}{1}}{\frac{2}{3} \times \frac{15}{1}} = \frac{12}{10}$$

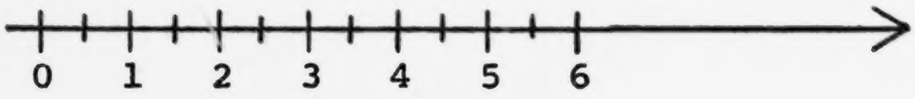
2) $\frac{4}{5} \div \frac{2}{3}$

Rewrite as a complex fraction: $\dfrac{\frac{4}{5}}{\frac{2}{3}}$

Multiply by identify element expressed as the reciprocal of the denominator over itself:

$$\frac{\frac{4}{5}}{\frac{2}{3}} \times \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{12}{10}}{\frac{6}{6}} = \frac{\frac{12}{10}}{1} = \frac{12}{10}$$

3) The number line can be used to illustrate division of fractions in special cases.



Example: $5 \div \frac{1}{2}$

Divide 5 sections of the number line into halves. Count the number of halves. If you do not wish to count the number of halves, you can find the answer by using this method. There are 5 groups of 2 halves or a total of 10 parts. Therefore, $5 \div \frac{1}{2}$ is the same as $5 \times \frac{2}{1}$ which is 10.

Thus, by using one of the above methods for division of fractions, we can show that division of fractions is the same as multiplying by the inverse of the divisor in the special case.

C. Activities which can be used to review the basic operations with rational numbers.

1. Discover the pattern for adding these fractions:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{9 \times 10} =$$

(It isn't necessary to do this by adding $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{90}$ by the long method.)

Look at the first term $\left(\frac{1}{2}\right)$

Add the first and second terms $\left(\frac{2}{3} \text{ is the result}\right)$

Add the first, second and third terms $\left(\frac{3}{4} \text{ is the result}\right)$

What appears to be true? (Sum of any number of terms will be a fraction formed by using the two factors of the nth term's denominator to form a fraction.)

Check by adding: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} = \frac{4}{5}$

What is the final result of the original question? $\left(\frac{9}{10}\right)$

2. Directions: Insert operation and grouping symbols that will make each statement true.

a. $\frac{1}{2} \square \frac{1}{2} \square \frac{1}{2} = \frac{1}{2}$

b. $\frac{1}{2} \square \frac{1}{2} \square \frac{1}{2} = \frac{3}{2}$

c. $\frac{1}{2} \square \frac{1}{2} \square \frac{1}{2} = \frac{1}{8}$

d. $\frac{1}{2} \square \frac{1}{2} \square \frac{1}{2} = 2$

e. $\frac{1}{2} \square \frac{1}{2} \square \frac{1}{2} = -\frac{1}{2}$

The teacher may or may not wish to show the students the relationships that exist between these patterns and the rational number operations.

D. Decimal Fractions.

1. Multiplication of decimal fractions:

Problem	Computation	Fractional Form	Decimal Form
23 x 47	$\frac{23}{1} \times \frac{47}{1}$	$\frac{1081}{1}$	1081.
2.3 x 47	$\frac{23}{10} \times \frac{47}{1}$	$\frac{1081}{10}$	108.1
2.3 x 4.7	$\frac{23}{10} \times \frac{47}{10}$	$\frac{1081}{100}$	10.81
.23 x 4.7	$\frac{23}{100} \times \frac{47}{10}$	$\frac{1081}{1000}$	1.081
.23 x .47	$\frac{23}{100} \times \frac{47}{100}$	$\frac{1081}{10000}$	.1081

Have students look for patterns when such values as these are multiplied.

a. In fractional form the numerator is always the same, which emphasizes the fact that multiplication of decimal fractions is similar to multiplication of whole numbers.

b. Observe that the denominators of the fractional forms tell us the place values; therefore, we can write the fractional form as a decimal fractional numeral.

c. Complete the following:

1) .1 x 1 = _____

2) .1 x .1 = _____

3) .1 x .01 = _____

2. Division of decimal fractions.

Example: Find the quotient of this problem.

$$1.2\overline{)\,1.44}$$

a. Rewrite the problem in the form of a fractional numeral. $\frac{1.44}{1.2}$

b. Since this is a fractional numeral, it can be multiplied by the identity element for multiplication and its value will not change.

$$\frac{1.44}{1.2} \times 1$$

c. Rename the identity element in such a way that when the denominator is multiplied by the denominator of the identity it will become an integer. $\frac{1.44}{1.2} \times \frac{10}{10} = \frac{14.4}{12}$

d. Rewrite the problem in original form and perform the division.

$$\begin{array}{r} 1.2 \\ 12\,\overline{)\,14.4} \\ \underline{12} \\ 2.4 \\ \underline{2.4} \end{array}$$

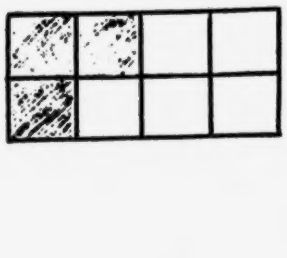
e. The decimal is placed in the answer by estimation.

E. An Approach to Percentage.

1. Meanings of fractions.

a. We use a fraction to show a part, or parts of a whole.

Example:



Drawing shows the shaded area is $\frac{3}{8}$ of the whole rectangle.

The denominator is the lower portion of a fraction. It is the number which tells the size of the units. The numerator is the number above the line. It tells the number of unit parts which are to be considered.

b. A second meaning represented by a fraction is a division relation. Thus, $\frac{6}{2}$ means $6 \div 2 = 3$, $9 \div 2 = \frac{9}{2} = 4 + \frac{1}{2}$, etc. If 8 apples cost 56 cents and we know that each of them costs the same, then we can find the cost of one apple. We say $\frac{56}{8} = 56 \div 8 = 7$. One apple costs 7 cents. We use $\frac{56}{8}$ to show division.

c. The third use of fractions is to show comparison, as in a ratio. A ratio is the quotient, or fraction relation, of the measures of two like quantities. The measures must be in terms of the same unit. Therefore, the ratio of 3 ft. to 12 ft. is $\frac{3}{12}$, or $\frac{1}{4}$, or $3 \div 12$. Likewise, the ratio of \$1.00 to \$10.00 is $\frac{1}{10}$.

We have thus shown three meanings for fractions:

1) To show a part or parts of a whole.
2) To show a division relation.
3) To compare like measures, as in a ratio.

2. If pencils are sold at the rate of two for five cents, how many pencils can you buy for twenty-five cents?

   a. One way to express a ratio such as "two for five cents" is $\frac{2\text{(two pencils)}}{5\text{ (five cents)}}$.

   b. There is some number of pencils that we can buy for 25 cents. If n designates the number of pencils we can buy for 25 cents, then we can write the ratio $\frac{n}{25}$.

   c. The relationship between $\frac{2}{5}$ and $\frac{n}{25}$ is one of equality and can be expressed as the proportion $\frac{2}{5} = \frac{n}{25}$.

   d. If $\frac{2}{5} = \frac{n}{25}$ then the numerator and denominator of $\frac{2}{5}$ must each be multiplied by 5. This is the same as multiplying the ratio by 1, the identity element. The value of n is therefore 2 x 5 or 10.

3. The three traditional percentage type problems are made more meaningful by using a ratio-proportion approach.

   Traditional:

   a. 30% of 10 = _____

   b. _____% of 10 = 3

   c. 30% of _____ = 3.

Ratio-proportion approach for the same three cases:

a. 30% means $\frac{30}{100}$ therefore $\frac{30}{100} = \frac{n}{10}$ or $30 \times 10 = 100 \times n$.

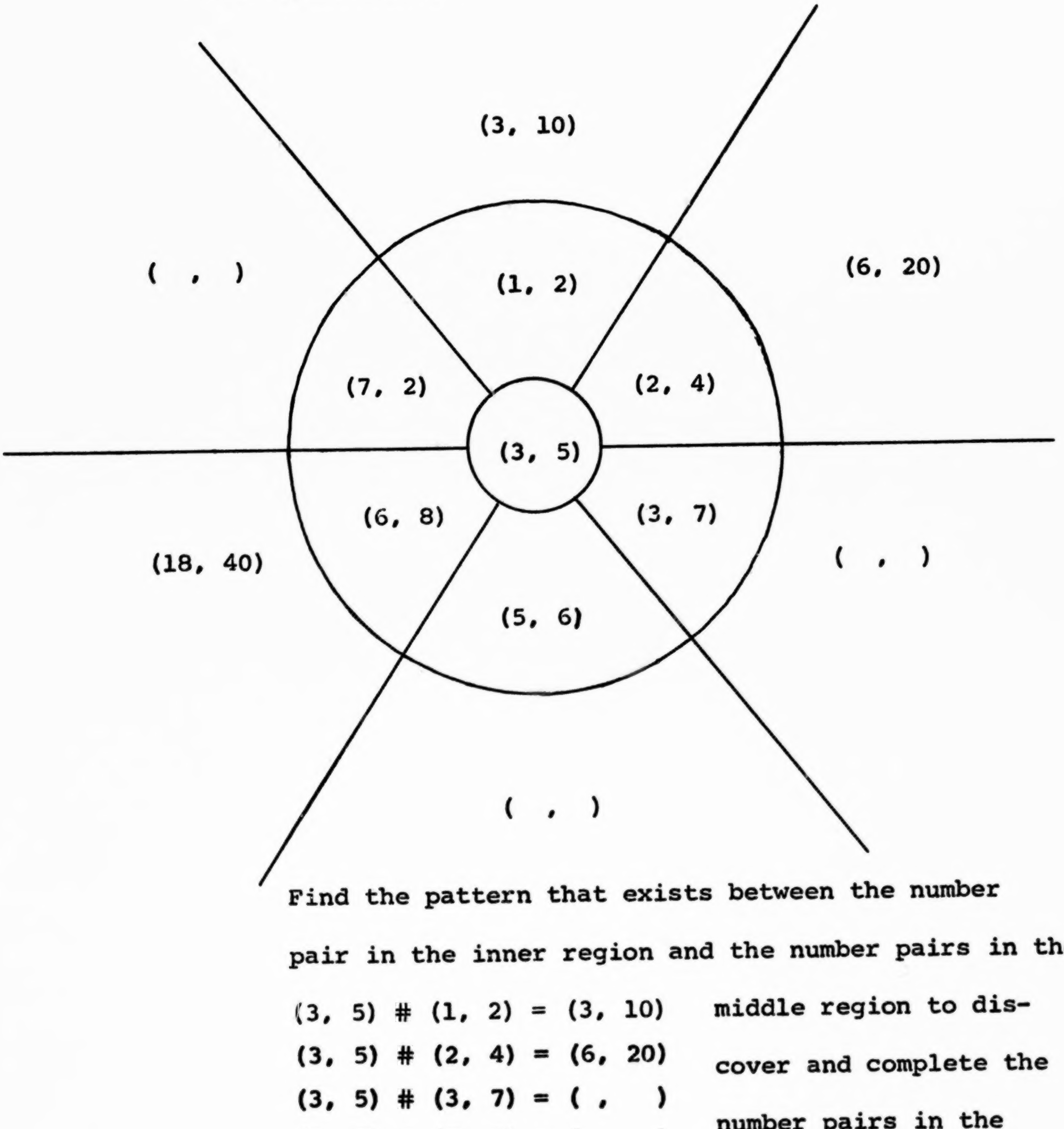
b. Since the percent is unknown, this problem can be expressed as $\frac{n}{100} = \frac{3}{10}$ or $n \times 10 = 100 \times 3$.

c. To find the unknown in this case, express the problem as $\frac{3}{n} = \frac{30}{100}$ or $3 \times 100 = n \times 30$.

NOTE: The three types of percentage problems traditionally taught can also be treated by a number sentence approach. By this method they reduce to one case.

4. Abstract pattern approach to operations with rational numbers.

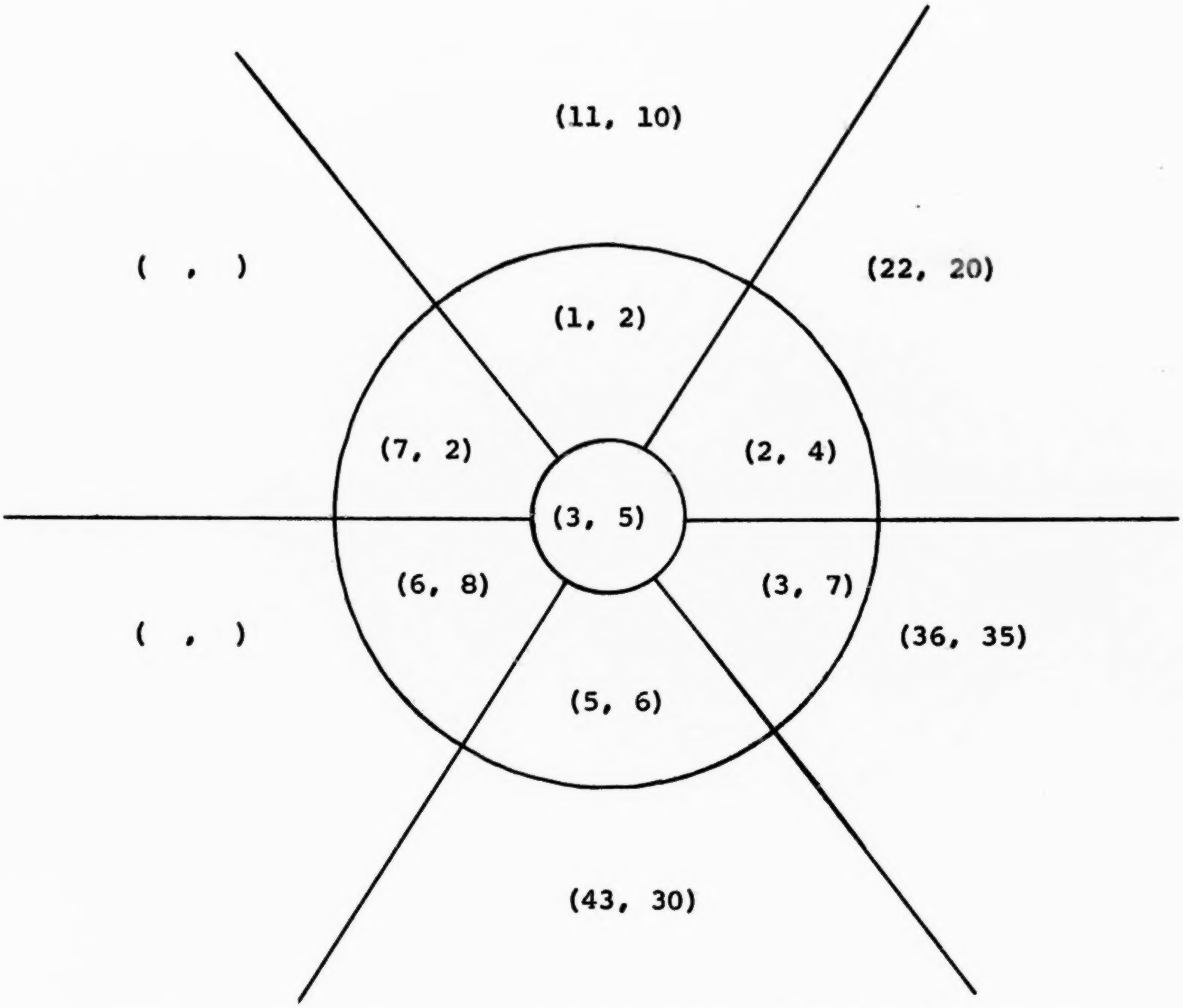
a. Operation # .



Find the pattern that exists between the number pair in the inner region and the number pairs in the middle region to discover and complete the number pairs in the outer region.

(3, 5) # (1, 2) = (3, 10)
(3, 5) # (2, 4) = (6, 20)
(3, 5) # (3, 7) = (,)
(3, 5) # (5, 6) = (,)
(3, 5) # (6, 8) = (18,40)
(3, 5) # (7, 2) = (,)

b. Operation *.


(11, 10)
(,)
(22, 20)
(1, 2)
(7, 2)
(2, 4)
(3, 5)
(6, 8)
(3, 7)
(,)
(36, 35)
(5, 6)
(43, 30)

Find the pattern that exists between the number pair in the inner region and the number pairs in the middle region to discover and complete the number pairs in the outer region.

(3, 5) * (1, 2) = (11, 10)
(3, 5) * (2, 4) = (22, 20)
(3, 5) * (3, 7) = (36, 35)
(3, 5) * (5, 6) = (43, 30)
(3, 5) * (6, 8) = (,)
(3, 5) * (7, 2) = (,)

5. Use of a cross number puzzle can also make this topic more interesting.

Percent Crossnumber Puzzle

1	2	▨	3	4	5
6		▨	▨	7	
▨	8		▨		▨
10	▨	▨	11		12
13	14	▨	15	▨	
▨	16		▨	17	▨

Horizontal

1. 50% of 86 = what number?
3. 80% of 565 = what number?
6. 7 = $12\frac{1}{2}\%$ of what number?
7. A bicycle cost \$48. It was sold at a $33\frac{1}{3}\%$ loss. Find the selling price.
8. A grade of 75% was increased by 12%. What is the new grade in %?
11. A boat reduced $16\frac{2}{3}\%$ from \$366 is sold for how much?
13. What number, decreased by $37\frac{1}{2}\%$ of itself, is 50?
16. During a chess tournament Jim won 47 out of 50 games. What % of the games played did Jim win?
17. $\frac{1}{2}\%$ of 1000 = what number?

Vertical

1. 30 = $66\frac{2}{3}\%$ of what number?
2. 320 increased by 15% is what number?
4. A \$5800 salary is decreased by 10%. Find the new salary.
5. 22 is what % of 100?
10. Sam learned 39 out of 50 words. What % did he learn?
11. A team won 13 ball games and lost 7. What % did it lose?
12. $\frac{1}{3}\%$ of 15000 = what number?
14. Write 9% as a decimal fractional numeral.

^{1}4	^{2}3	▨	^{3}4	^{4}5	^{5}2
^{6}5	6	▨	▨	^{7}3	2
▨	^{8}8	7	▨	2	▨
^{10}7	▨	▨	^{11}3	0	^{12}5
^{13}8	^{14}0	▨	^{15}5	▨	0
▨	^{16}9	4	▨	^{17}5	▨

CHAPTER V

MISCELLANEOUS TOPICS

Suggestions for Teaching Place Value Using Exponential Notation

1. Use 10 as a repeated factor, developing the powers of 10:

   $10 = 10 = 10^1$

   $100 = 10 \times 10 = 10^2$

   $1,000 = 10 \times 10 \times 10 = 10^3$

   $10,000 = 10 \times 10 \times 10 \times 10 = 10^4$

   $100,000 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$

   $1,000,000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$

   NOTE: Student should discover the pattern that the exponent corresponds with the number of zeros after the digit 1.

2. 6,000, 30,000, and 400,000 may be expressed as:

   $6 \times 1,000 = 6 \times 10^3$

   $3 \times 10,000 = 3 \times 10^4$

   $4 \times 100,000 = 4 \times 10^5$

3. Place Value Chart--exercise such as this can be completed by the student:

10^6 M.	10^5 H. Ths.	10^4 T. Ths.	10^3 Ths.	10^2 Hund.	10^1 Tens	10^0 Ones
			4	5	6	5

$= 4000 + 500 + 60 + 5$

$= (4 \times 1000) + 5(100) + 6(10) + 5$

$= 4(10^3) + 5(10^2) + 6(10^1) + 5$

$= 4{,}565$

4. The mean distance between Earth and Sun is about 93,000,000 miles.

   How many ways can this be written?

   Answer: 93,000,000

   93 million

   $93 \times 1{,}000{,}000$

   93×10^6

   9.3×10^7 (Use only if appropriate for your class.)

5. Write as an expanded numeral:

   3 million, 4 hundred thousand, 9 ten thousand, 0 thousand, 5 hundreds, 8 tens, 2 ones.

   Answer: $3 \times 10^6 + 4 \times 10^5 + 9 \times 10^4 + 5 \times 10^2$

   $8 \times 10^1 + 2 \times 10^0$

6. Regrouping:

   In the numeral 3,525:

   a. Write as an expanded numeral.

   $3(10^3) + 5(10^2) + 2(10^1) + 5(10^0)$

b. Show how many groups of 100 are contained in this number.

$\underline{35(10^2)} + 2(10^1) + 5(10^0)$

c. Show how many groups of ten are contained in this number.

$\underline{352(10^1)} + 5(10^0)$

7. Express the following numbers in powers of 10:

$71 = 7(10^1) + 1(10^0)$

$234 = 2(10^2) + 3(10^1) + 4(10^0)$

$6{,}108 = 6(10^3) + 1(10^2) + 8(10^0)$

$200{,}000{,}010 =$

$606{,}006{,}666 =$

8. Addition of whole numbers using exponential notation (show principles) and renaming using expanded numerals.

$36 + 42 =$

$3(10^1) + 6(10^0) + 4(10^1) + 2(10^0)$

$3(10) + 6(1) + 4(10) + 2(1)$ (renaming property)

$3(10) + 4(10) + 6(1) + 2(1)$ (commutative)

$(3 + 4)\ (10) + (6 + 2)\ (1)$ (distributive)

$7(10) + 8(1)$ (definition of addition)

78

Other Numeration Systems

Since our numeration system is based on ten and multiples of 10, we can refer to a number system which is based on 5 and multiples of 5 as a Base Five Numeration System.

The purpose of teaching the base five numeration system is to reinforce and make more meaningful the base ten numeration system.

Comparison of base ten and base five.

Base ten: There are 9 digits and zero.

0,1,2,3,4,5,6,7,8,9

Base five: There are 4 digits and zero.

0,1,2,3,4

Place value chart for base ten:

10^3	10^2	10^1	10^0
Thousands	Hundreds	Tens	Ones

Place value chart for base five:

5^3	5^2	5^1	5^0
One hundred twenty-fives	Twenty-fives	Fives	Ones

By placing digits in the place value chart for base ten in this manner:

100's	10's	1's
1	2	3

We read this number as one group of 100's; 2 groups of 10's; 3 groups of 1's. By placing the same three digits in a base five place value chart:

25's	5's	1's
1	2	3

We read the number as one group of 25's; 2 groups of 5's; 3 groups of 1's. This type of place value chart can be extended to include any numeration system which you choose to teach, such as base 2, 4, 7, 12, etc.

Suggestions

1. Use of abacus for easy computation in new base.
2. Addition table in base five.

   Patterns can be discovered by students in the same manner as in base ten.

+	1	2	3	4	10
1	2	3	4	10	11
2	3	4	10	11	12
3	4	10	11	12	13
4	10	11	12	13	14
10	11	12	13	14	20

3. Multiplication table in base five:

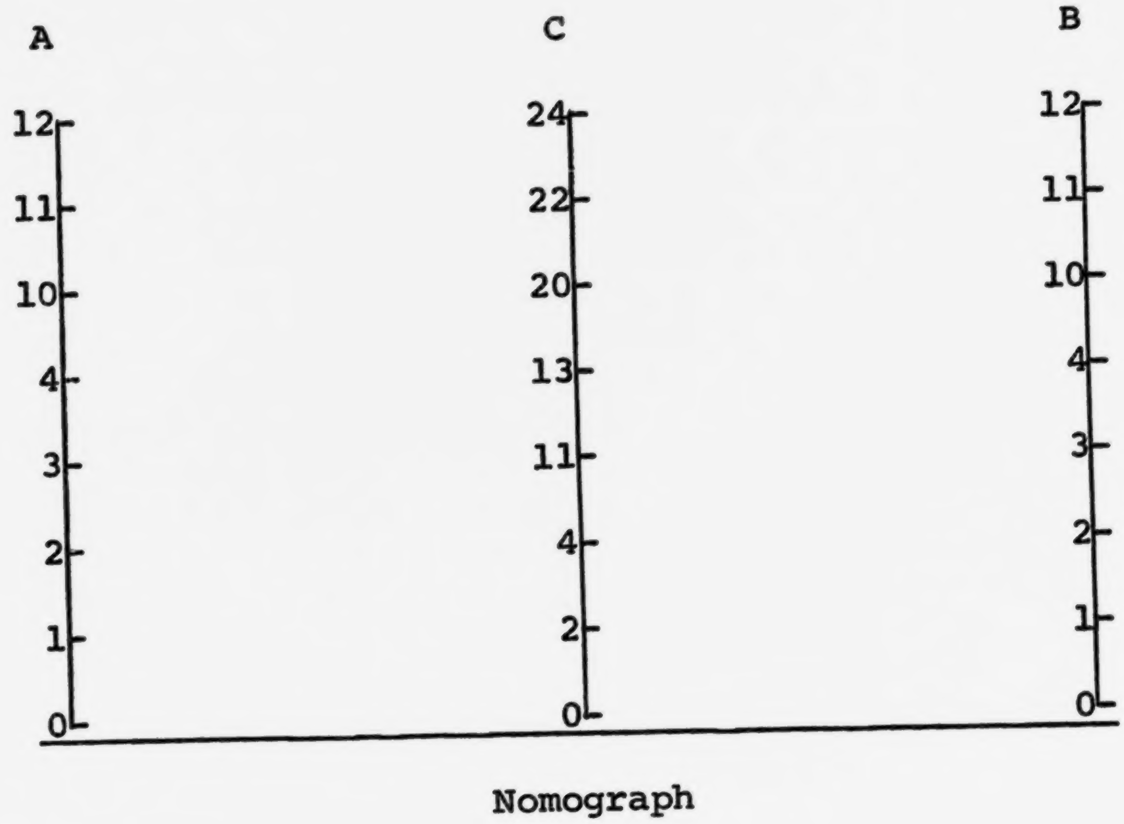
x	1	2	3	4	10
1	1	2	3	4	10
2	2	4	11	13	20
3	3	11	14	22	30
4	4	13	22	31	40
10	10	20	30	40	100

4. Nomograph for base five: (See below)

5. Ask the students to obtain a given number such as 44, by placing the fewest number of marks in a chart as pictured below

<table>
<tr><td>25</td><td>5</td><td>1</td></tr>
<tr><td>|</td><td>|||</td><td>||||</td></tr>
</table>

$134_{five} = 44_{ten}$


A
C
B
12
11
10
4
3
2
1
0
24
22
20
13
11
4
2
0
12
11
10
4
3
2
1
0

Nomograph

Activities for Developing the Concepts of Probability

In order to give the class an idea of probability, the following activities are suggested:

A. How many heads or tails?

1. Have each student remove a coin from his pocket or purse and place it on his desk keeping it covered at all times. The student should not look at the coin.
2. Ask each student to guess whether he thinks his coin is a head or tail. The teacher should chart the guesses on the board or overhead projector.
3. Students are asked to look at their coins and a chart can be made to compare the guesses with the actual results.
4. After repeating this activity several times, the students should discover that the number of heads and tails will be approximately equal.

B. Two experiments.

1. Experiment 1:

The Southeastern High School baseball team has played 15 games this season. The record for three of the players is as follows:

Player	Times at Bat	Hits
Centerfielder	60	27
Shortstop	50	18
First Baseman	45	14

What is the probability that the centerfielder gets a hit the next time at bat? (27/60 = .450)

Shortstop: (18/50 = .360)

First Baseman: (14/45 = .311)

2. Experiment 2:

In a mathematics class there are 18 boys and 12 girls. The names of all the class members are placed on cards in a box, mixed up thoroughly, and a card is drawn without looking.

What are the chances that the name drawn will be a girl? (12/30 or 2/5)

Finger Computation--Multiplication

Multiplication by 9

1. If fingers of both hands are assigned values as shown on Figure 1, multiplication of 9 by any whole number up to 10 can be performed.

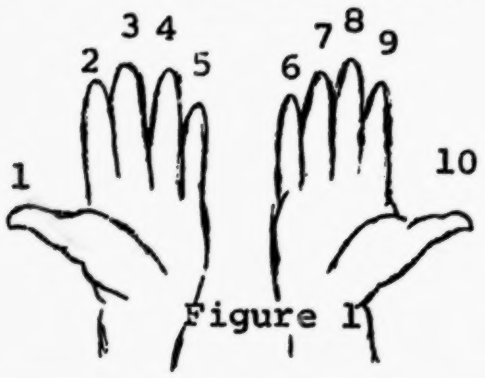


Figure 1

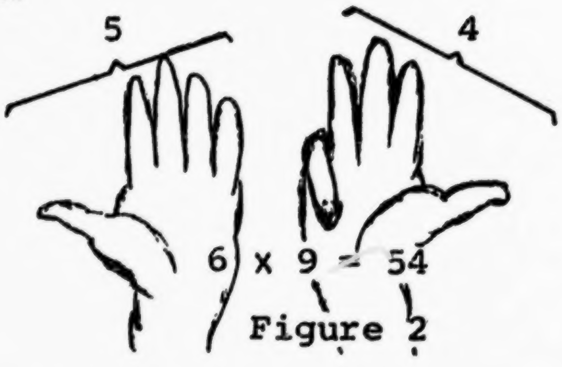
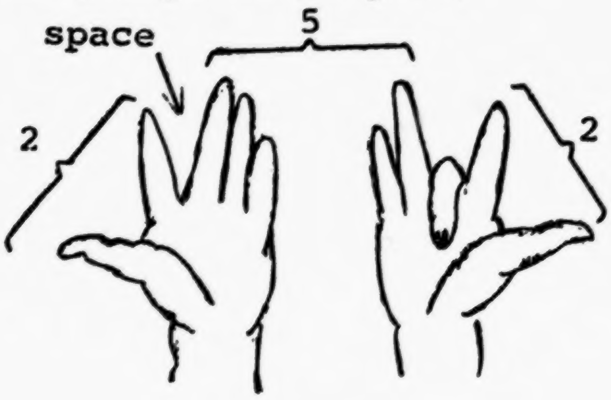


Figure 2

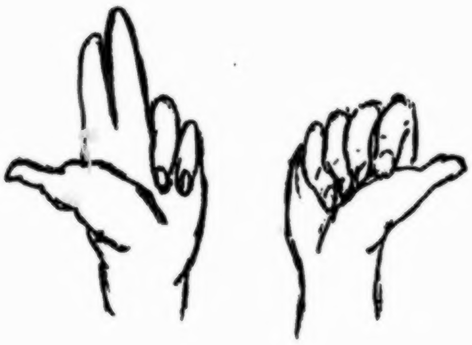
To multiply 9 x 6, turn finger 6 (Figure 2). The five fingers to the left and the four to the right give the answer 54.

2. The product of 9 and any two-digit number can be found provided the ten's digit of the two-digit number is less than its one's digit. To multiply 9 x 28, create a space after the second finger from the left (for the ten's digit) and put the eighth finger down (for the one's digit). Read your product in terms of groups of fingers.



9 x 28 = 252

3. Another multiplication involves any two numbers between 5 and 10 such as 8 x 6. The closed fist represents five, one finger raised is 6, two raised is 7, 3 raised is 8, 4 raised is 9, and 5 raised is 10. To multiply 8 x 6, show 8 on one hand and 6 on the other. To find the number of tens in the product, add the extended fingers. To find the number of ones in the product, multiply the number of down fingers on one hand by the number of down fingers on the other hand.



Extended fingers: 3 + 1 = 4

Down fingers: 2 x 4 = 8

Result: 8 x 6 = 48

More on Magic Squares

As pointed out in the Introduction, magic squares can be made from using consecutive numbers. These are not the only types of magic squares that exist. A magic square can be made from any pattern of numbers such as multiples of two, square numbers, or by transposing an existing magic square.

To transpose a magic square of 16 cells:

1. Construct a magic square of 16 cells as you have already learned to do in the Introduction (Figure 1).

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figure 1

2. Take two blank 16-celled squares and:
   a. Divide one into quarters and label as in Figure 2.
   b. Number the columns of the other one on the outside as shown in Figure 3.

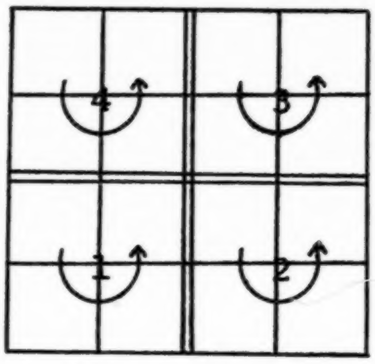


Figure 2

1	2	3	4
9	6	3	16
4	15	10	5
14	1	8	11
7	12	13	2

Figure 3

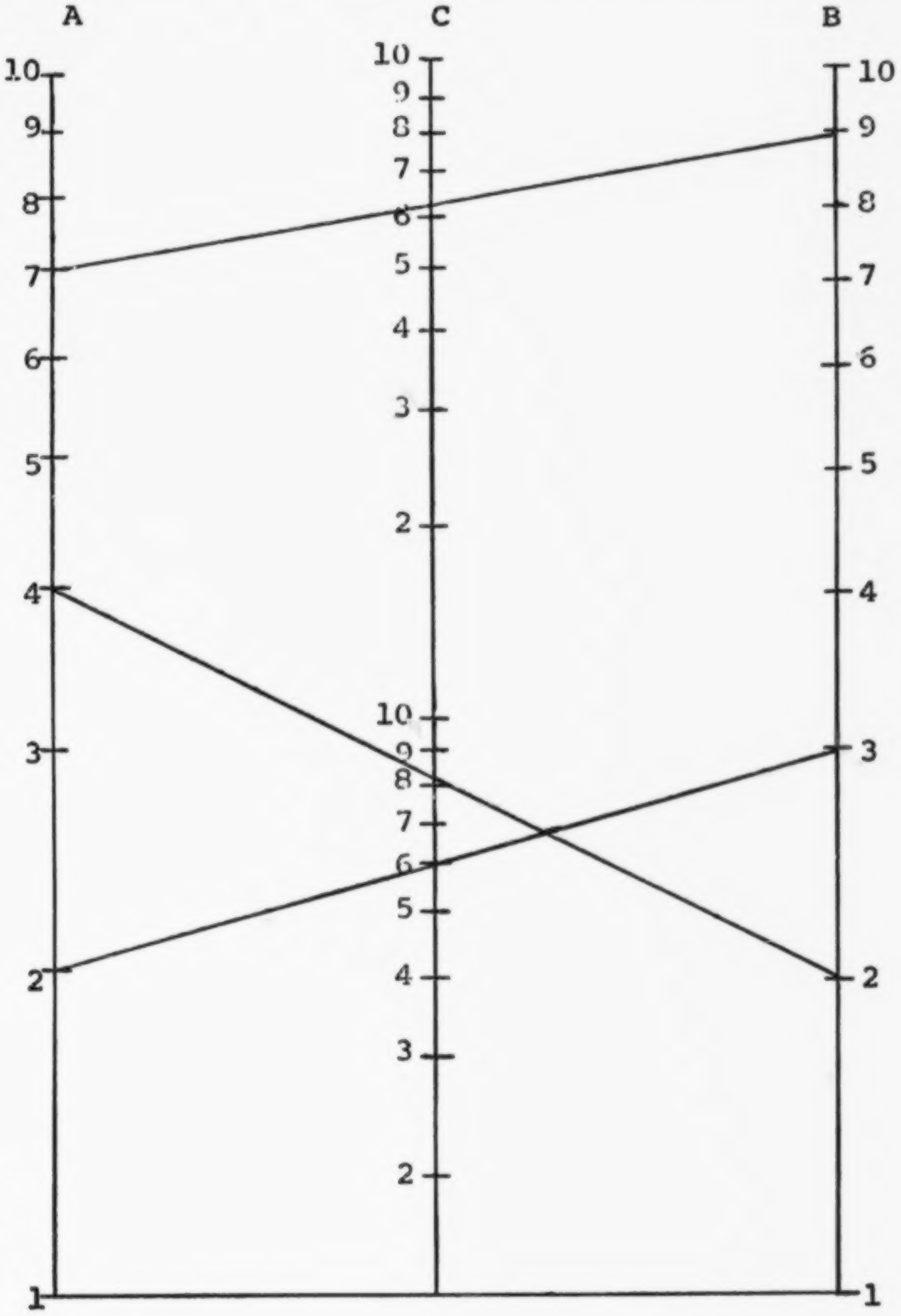
3. Column one corresponds to quarter one of Figure 2 and, by following the arrow, column one is transposed to

9
4
14
7

In like manner the other three columns can be transposed.

Nomograph for Multiplication and Division

To make a nomograph for multiplication and division, use logarithmic and semi-logarithmic graph paper or a slide rule. Start with three equally spaced vertical line segments of equal length and label them A, C, B respectively. Scales A and C are equivalent to the C and D scales on a slide rule. Scale B is equivalent to scale A on the slide rule or to the scale on a sheet of semi-logarithmic graph paper.



Directions for use are the same as those given for the addition nomograph.

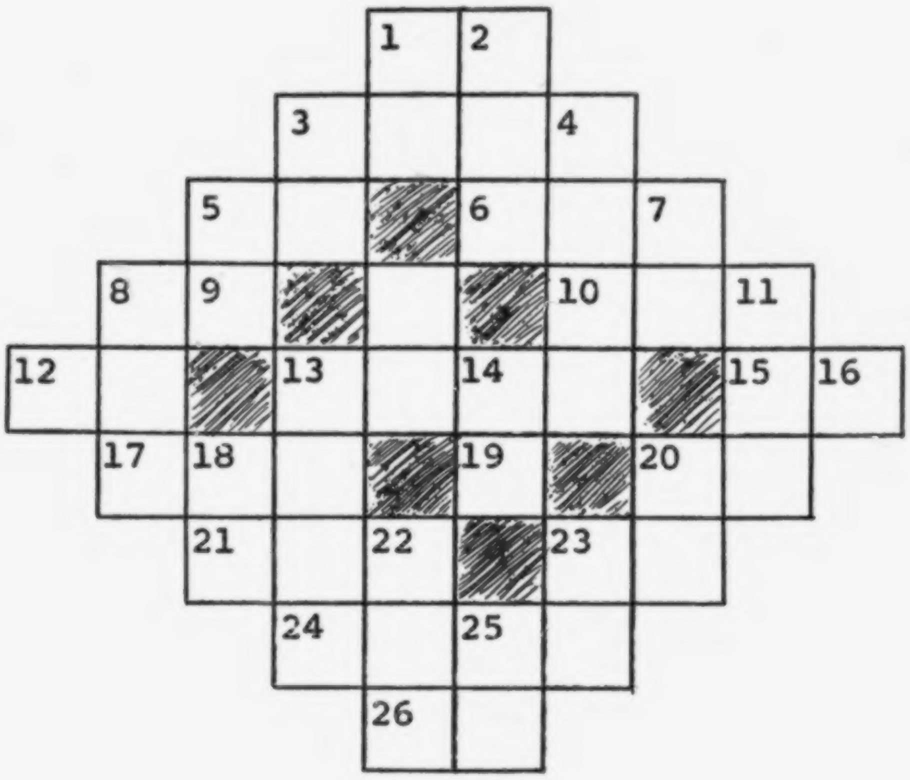
Try these examples:

7 x 9 = 63

4 x 2 = 8

6 ÷ 2 = 3

Review of Whole Numbers



Horizontal

1. 44 + 7 + 23 + 18.
3. Write as a number: Four thousand, four.
5. Round off 67 to the nearest ten.
6. 2,214 - 1,389.
8. Next consecutive odd whole number after 11.
9. 315 / 35.
10. 13 x 4 x 7.
12. Six ten's and a 7.
13. 62 x 68.
15. Find the sum of 47 and 34.
17. Number of hundred's contained in 15,000.

Vertical

1. Number of hundred's contained in 9,000.
2. 25 x 35 - 667.
3. 1,520 / 38.
4. 12 x 353.
5. 602 - 529.
7. $\frac{336 \times 135}{810}$
8. Find the sum of 76 and 95.
9. Average of 99; 85; 77; and 107.
11. Divide 2,840,601 by 5,809.
12. Number of ten's contained in 60.
13. Write as a number: Four thousand, twenty-six.
14. Round off 6 to the nearest ten.

Horizontal

19. Product obtained when a number is multiplied by zero.

20. The difference between 368 and 289.

21. Divide the product of 328 and 84 by 224.

23. Divide 24,492 by 314.

24. The product of 87 and 78.

26. Average of 9; 27; 20; 16; and 33.

Vertical

16. Quotient when a number is divided by itself.

18. First odd number after 50.

20. The product of 2; 3; and 13.

22. 5 + 48 + 195 + 9 + 78 + 37.

23. Difference between 4,063 and 3,987.

25. (7,452 / 108) + 12.

Review of Perimeters, Areas, and Volumes

1	2	▨	3	4	5	6
7		▨	8			
9		10	▨	11		▨
▨	▨	12	13		▨	▨
▨	14		▨	15	16	17
18			19	▨	20	
21				▨	22	

Horizontal

1. Perimeter of a 9-inch square in number of inches.
3. Area in number of square inches of a square with side $3\frac{1}{2}$ inches.
7. Circumference in number of inches of a circle having a diameter of 7 inches. (Use pi equals 22/7)
8. Area in number of square feet of a rectangle that is 17 inches by $14\frac{1}{2}$ inches.
9. The diameter of a circle is 14 inches. Find area in square inches. (Use pi equals 22/7.)
11. Perimeter in number of inches of a triangle 7 inches on a side.
12. Area in square inches of a triangle with a base of 19 inches and an altitude of 12 inches.

Vertical

1. Each side of a triangle is 107 inches. How many inches in the perimeter?
2. Number of square inches in a 25-inch square.
3. Number of inches in the perimeter of a 3-inch square.
4. Number of square feet in a parallelogram with an altitude of 44 feet and base of 551 feet.
5. Number of cubic inches in a metal plate 3 inches square and .029 inches thick.
6. Area in square inches of a parallelogram with an 11-inch base and a 5-inch altitude.
10. Number of gallons in a tank 37 feet by 15 feet by 10 feet. (Use 1 cubic foot = 7.5 gallons.)
13. Number of square inches in a 1-inch square.

(All fractions should be written as decimal fractions.)

Horizontal

14. Diameter in number of inches of a wheel having a radius of 8 inches.

15. Volume in number of cubic inches of a rectangular solid 9 inches by 8 inches by 6 inches.

18. Number of cubic inches in a cube 1 foot on a side.

20. How many 2-inch squares will fit into a rectangle 10 inches long and 6 inches wide?

21. Number of cubic inches in a right cylinder 10 inches in diameter and 10 inches high. (Use 3.1416 for pi.)

22. Perimeter in number of inches of a rectangle 15 inches long and 10 inches wide.

Vertical

14. Number of feet in the perimeter of a rectangle 65 feet long and 24 feet wide.

16. Number of cubic inches in a rectangular solid 9 inches long, 7 inches wide, and 5 inches high.

17. Number of cubic feet in a pyramid with a base 10 feet square and height of $7\frac{1}{2}$ feet.

18. Number of square yards in a triangle with a base of 18 feet and an altitude of 17 feet.

19. Number of cubic feet in a rectangular solid 7 feet long, 4 feet wide, and 3 feet deep.

Review of Fractions

▨	1	2	▨	▨	3	4
5		▨	6	7	▨	8
9		▨	10		11	▨
▨	▨	12	▨	13	▨	▨
▨	14		15	▨	16	17
18	▨	19		▨	20	
21	22	▨	▨	23		▨

Horizontal

1. The L.C.D. of $\frac{7}{8}$ and $\frac{5}{6}$.
3. Change $\frac{5}{8}$ to number of 24ths.
5. Change $\frac{36}{3}$ to a whole number.
6. $6\frac{2}{3} + 5\frac{1}{4} \div 3\frac{5}{6} + 2\frac{1}{4}$.
8. $7\frac{1}{2} - 3\frac{1}{2}$.
9. $8\frac{2}{5} \times 1\frac{2}{3}$.
10. $\frac{1}{4}$ of what number is 28?
12. $46\frac{1}{2} - 7\frac{3}{4}$.
13. Subtract $7\frac{1}{3}$ from $13\frac{1}{4}$ and add $2\frac{1}{12}$.
14. Add: $27\frac{5}{8}$; $32\frac{1}{3}$; $16\frac{5}{24}$; and $44\frac{5}{6}$.
16. Multiply $13\frac{5}{7}$ by $2\frac{1}{3}$.
18. Divide $29\frac{1}{2}$ by $7\frac{3}{8}$.

Vertical

1. Divide 14 by $\frac{1}{16}$.
2. Change $\frac{1}{4}$ to number of 16ths.
3. $\frac{1}{2} \div \frac{1}{6} + \frac{1}{3}$.
4. Find $\frac{3}{4}$ of 72.
5. $9\frac{1}{6} \times 1\frac{1}{5}$.
6. Divide $3\frac{3}{10}$ by $\frac{3}{10}$.
7. Add: $148\frac{3}{4}$; $210\frac{2}{3}$; $56\frac{1}{6}$; and $402\frac{5}{12}$.
11. Change $\frac{16}{8}$ to a whole number.
12. Multiply the difference of $120\frac{4}{5}$ and $27\frac{1}{2}$ by $6\frac{2}{3}$.
14. $\frac{2}{5} \times 1\frac{1}{2} \times 1\frac{2}{3}$.
15. 12 is $\frac{3}{4}$ of what number?
16. Take $\frac{4}{5}$ of 390.
17. L.C.D. of $\frac{1}{8}$; $\frac{1}{3}$; $\frac{3}{4}$; and $\frac{5}{12}$.

Horizontal

19. $19\frac{1}{2}$ is $\frac{3}{4}$ of what number?

20. $5\frac{3}{5} \times 1\frac{1}{2}$ divided by $\frac{3}{5}$.

21. Simplify:

$$\frac{12\frac{3}{5} \times 2\frac{1}{2}}{1\frac{1}{2}}$$

23. Divide the product of $17\frac{1}{2}$ and $\frac{2}{5}$ by $\frac{7}{12}$

Vertical

18. $\frac{3}{4}$ of what number is $31\frac{1}{2}$?

22. $1\frac{1}{5} \times 2\frac{1}{2}$ divided by 3.

23. Simplify:

$$\frac{5\frac{1}{2} - 2\frac{1}{4}}{1\frac{3}{4} + 1\frac{1}{2}}$$

Solutions for Puzzles

				9	2				
			4	0	0	4			
		7	0	▨	8	2	5		
	1	3	▨	9	▨	3	6	4	
6	7	▨	4	2	1	6	▨	8	1
	1	5	0	▨	0	▨	7	9	
		1	2	3	▨	7	8		
			6	7	8	6			
				2	1				

Review of Whole Numbers

3	6	▨	1	2	2	5
2	2	▨	2	4	6	5
1	5	4	▨	2	1	▨
▨	▨	1	1	4	▨	▨
▨	1	6	▨	4	3	2
1	7	2	8	▨	1	5
7	8	5	4	▨	5	0

Review of Perimeters,
Areas, and Volumes

▨	2	4	▨	▨	1	5
1	2	▨	1	8	▨	4
1	4	▨	1	1	2	▨
▨	▨	6	▨	8	▨	▨
▨	1	2	1	▨	3	2
4	▨	2	6	▨	1	4
2	1	▨	▨	1	2	▨

Review of Fractions

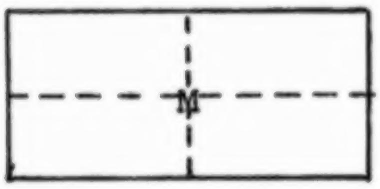
Paper Folding and Geometry

1. Point

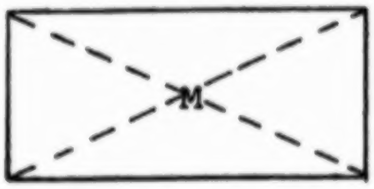
Ask each student to find the central point or position of a piece of paper (wax paper is excellent for this purpose) having square corners without using any instruments.

The two possible solutions are:

a.



b.



Be sure to have students justify that M is the central point by measuring the indicated dotted lines. Does this central point have any dimension? How can we discover whether or not the central point has any size or shape? A good exercise would be to demonstrate to the class by cutting the piece of paper on the folds and then placing the four pieces back in the original position. Emphasize that the place where the pieces meet is called a point; it has no dimension but it does have a fixed position or location.

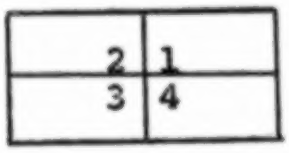
The idea of a geometric point may be illustrated further by marking a large dot on the board, erasing it, and replacing it with a smaller dot in the same position. Finally, erase the dot without replacing it and ask if the point remains. Thinking of a dot as shrinking until all its dimensions vanish gives one a good concept of the meaning of a geometric point. It indicates position only. The dot is merely a symbol.

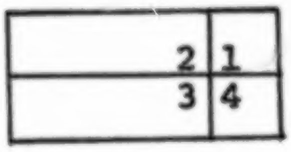
Tossing a Ball: Toss a ball straight up into the air, then ask: At what point did the ball start down? Does that point have any size? Emphasize the fact that this point has location but no size.

2. Lines

   a. A single crease in the paper represents a set of points or a straight line.

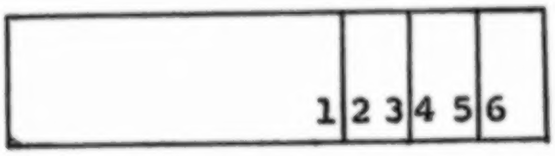
   b. Fold the paper so the side edges are together. Fold again so that the creased edges are together. Resulting creases produce perpendicular lines.





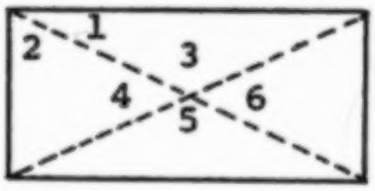
   Use measurement of indicated angles with the protractor to discover what must be true concerning perpendicular lines. (Right angles.)

   c. Fold the paper so that the side edges are together. Make another fold in the same direction keeping edges together. Resulting creases produce parallel lines. By measuring angles students



will discover that (1) lines perpendicular to the same line are parallel; (2) lines parallel to the same line are parallel to each other.

d. Fold the paper (with squared corners) on the diagonals. Measure all angles and segments in the figure. What are some of the things that can be discovered by the students?



1) Sum of the angles of a triangle equals 180°.

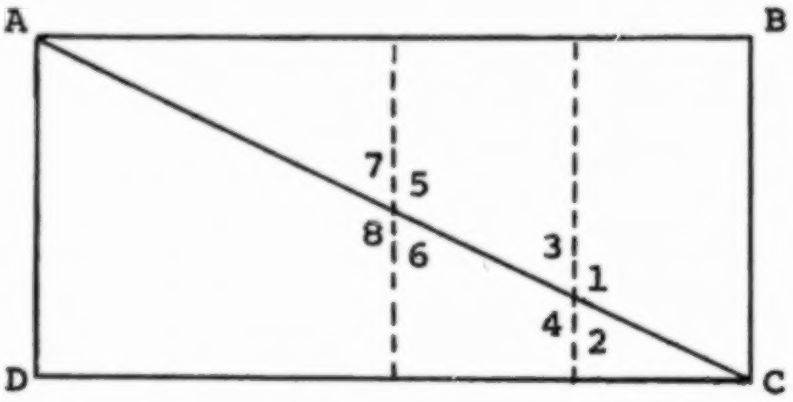
2) $\angle 1 + \angle 2 = 90^\circ$ (Complementary angles).

3) $\angle 3 + \angle 4 = 180^\circ$ (Supplementary angles).

4) $\angle 4 = \angle 6$ and $\angle 3 = \angle 5$ therefore opposite or vertical angles are equal.

5) Facts concerning congruent figures can be discovered.

e. Fold the paper on one diagonal. Open the sheet and fold so that the side edges are together. Make another fold in the same direction keeping edges together. Resulting creases show parallel lines intersected by a transversal.



Record measure of all angles:

Angle	1	2	3	4	5	6	7	8
Measure of Angle								

1) What have you discovered from the chart?

   Possible answers:

   a) Listing 2 sets of 4 equal angles.

   b) List equal angles by pairs.

   c) List supplementary angles by pairs.

2) Consider one pair of equal angles.

   a) If, for example, $\angle 1 = \angle 4$ is suggested, then discuss facts concerning vertical angles.

   b) If, for example, $\angle 1 = \angle 5$ is suggested, then discuss facts concerning corresponding angles and parallel lines.

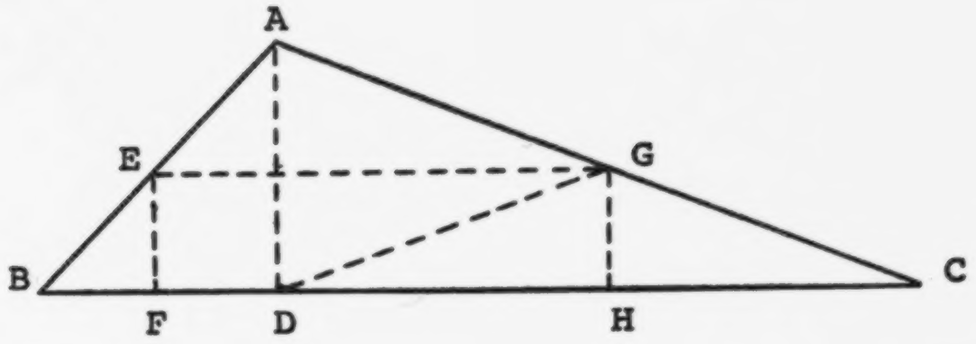
   c) If, for example, $\angle 3 = \angle 6$ is suggested, then discuss facts concerning alternate interior angles and parallel lines.

3) Consider pairs of supplementary angles.

   a) If, for example, $\angle 1$ and $\angle 2$ are given as supplements, then discuss property concerning adjacent angles whose exterior sides lie in a straight line.

   b) If, for example, $\angle 4$ and $\angle 6$ are given as supplements, then discuss property concerning interior angles on the same side of the transversal intersecting parallel lines.

Triangular Shape



1. Cut the interior of triangle ABC out of a piece of paper.
2. Fold B over the segment BC so that the crease passes through A. This determines point D on BC. (AD is the altitude of triangle ABC to side BC.)
3. Fold B over to D, forming crease EF. (Triangle BED is isosceles.)
4. Fold C over to D, forming crease GH. (Triangle DGC is isosceles.)
5. Fold A down to D, forming crease EG.
6. When all three angles A, B, and C are folded over to point D, they appear to form an angle of 180°. (The sum of the angles of a triangle has measure of 180°.)
7. Note that EF the altitude to the base of an isosceles triangle bisects the base BD, bisects the vertex angle BED and is perpendicular to the base (forms two equal, adjacent, supplementary angles).

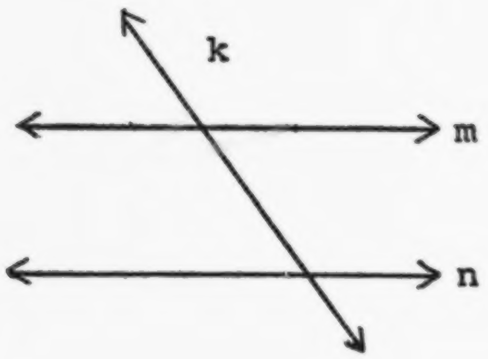
8. The length of EG, the line joining the mid-point of two sides of triangle ABC, is one-half the length of the third side, BC.
9. Note that the median DG to the hypotenuse of right triangle ADC is equal to one-half the hypotenuse.
10. When folded as directed in #6, note that the area of the original triangle is equal to twice the product of the altitude times the base of the rectangle formed.

Many other facts of geometry can be an outgrowth of this exercise.

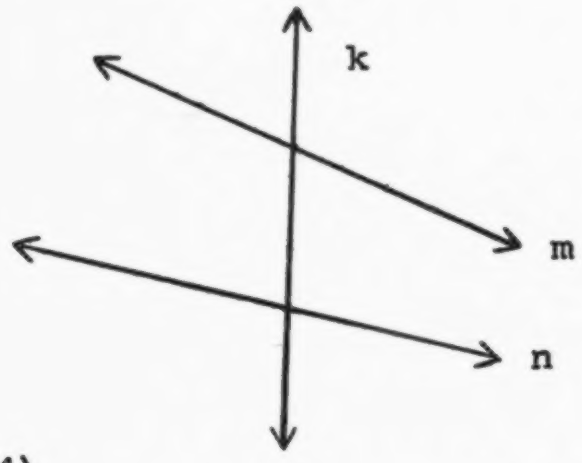
To Meet or Not to Meet?

Decide if m and n are parallel in each of these diagrams. Justify your answer for each diagram by using rulers and protractors.

(1)



(2)



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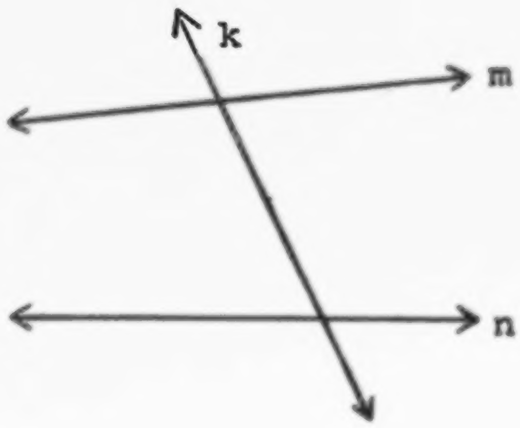

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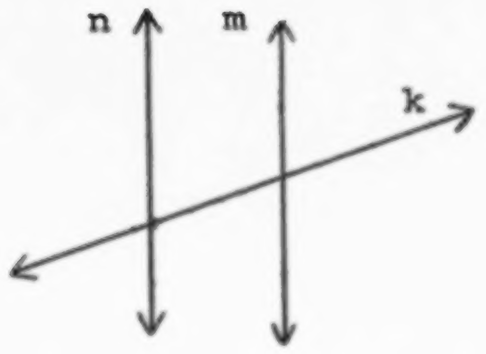

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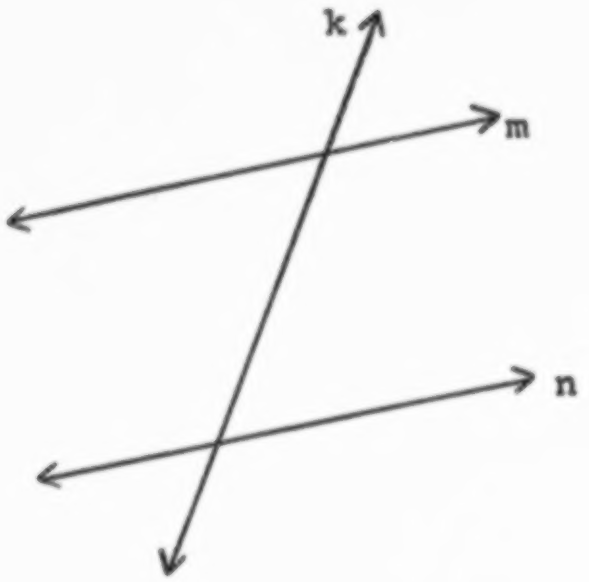

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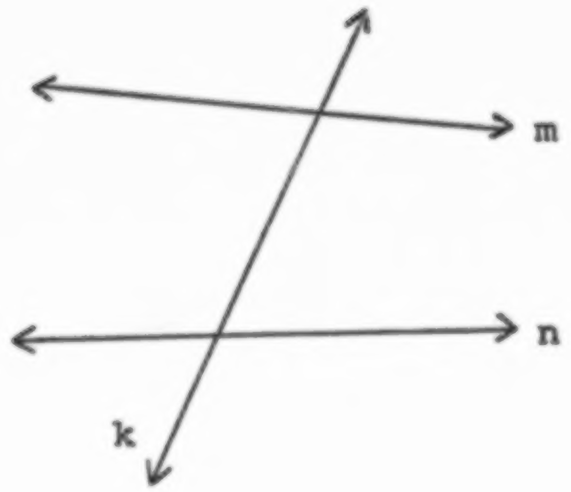
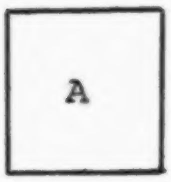
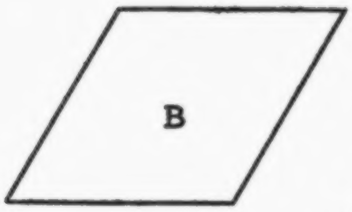
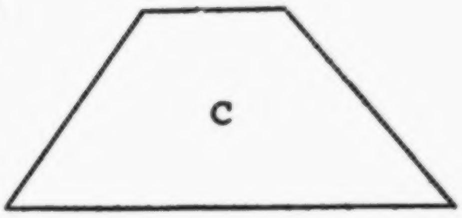
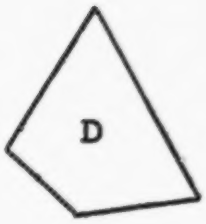


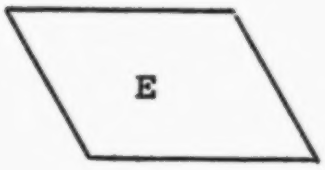
Quadrilaterals

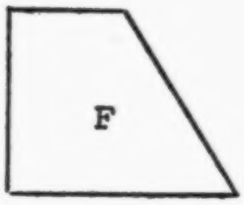


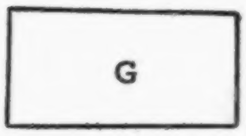


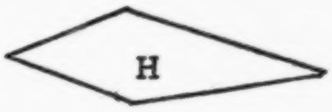










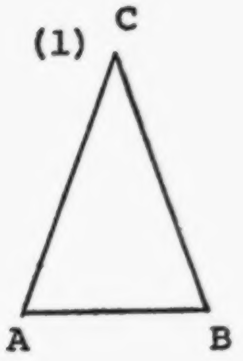


Make a check mark (✓) in the spaces below each lettered quadrilateral whenever the statement at the left is true of the given figures.

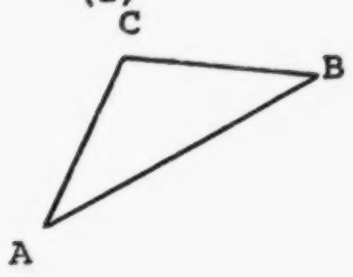
	A	B	C	D	E	F	G	H
All sides equal								
All angles equal								
Both pairs of opposite sides parallel								
All angles right angles								
Two sides parallel								
No sides parallel								
Name the figure								

Triangles

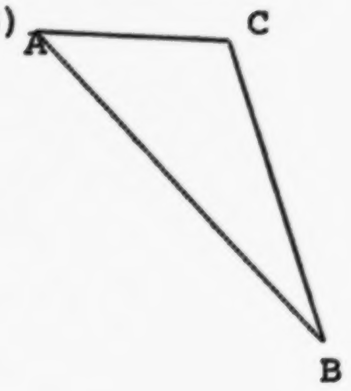
(1)



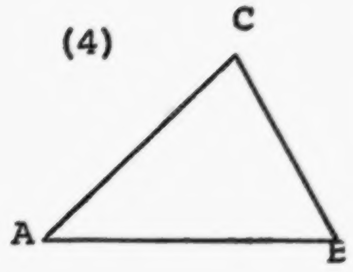
(2)



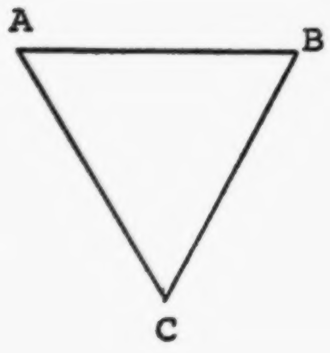
(3)



(4)



(5)



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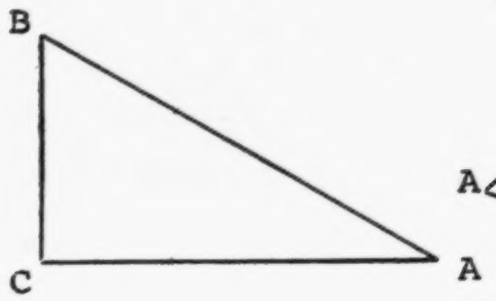

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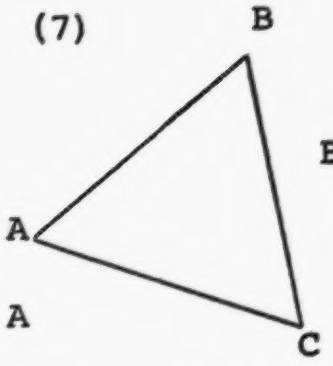

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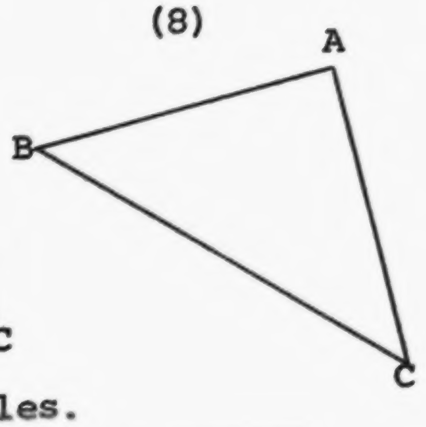


Measure all angles and sides of these triangles.

	1	2	3	4	5	6	7	8
∠A								
∠B								
∠C								
AB								
BC								
CA								

Are there any unusual features about these triangles?

Classify the triangles according to their angles and sides.

Elementary Surveying: Use of Clinometer to Determine an Angle

This activity could follow a review of either the concept of similar triangles or the concept of scale drawings.

Heights of flagpoles and buildings or rockets may be solved by the shadow method and similar triangles. Another interesting method is to use a Clinometer to determine the angle of elevation. Once the angle is determined, either scale drawing or trigonometry can be used to determine distances.

This activity gives every student the opportunity to discover surveying techniques for himself since he will make his own clinometer and use it.

1. Construction:
    a. Use rectangular piece of heavy cardboard, about 9 x 12 inches in size.
    b. Place a protractor close to the edge in the upper righthand corner, as shown in Figure 1. A 90° arc subdivided into degrees may be used.
    c. Attach a string with a weight at Point A, Figure 1.

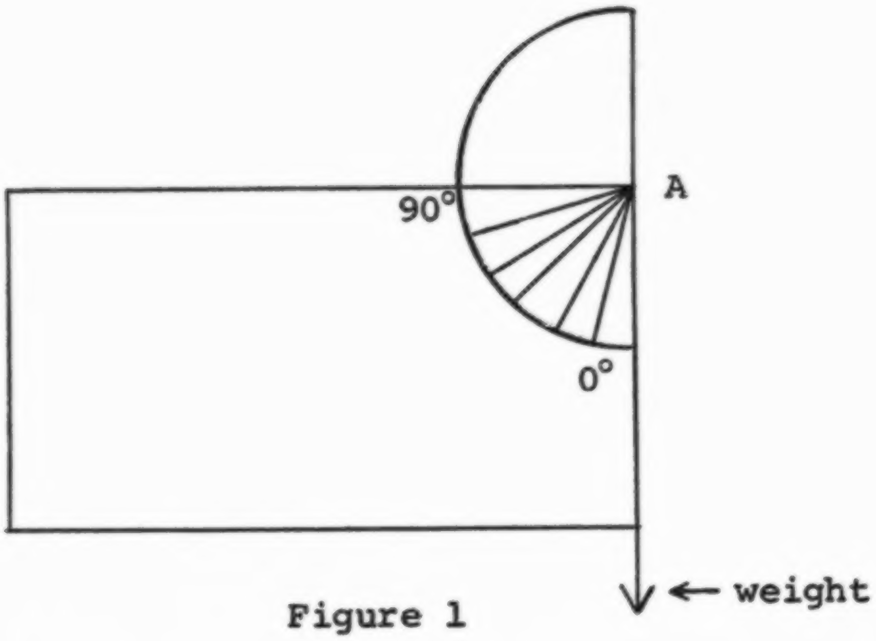

A
90°
0°
weight

Figure 1

2. Use:

   a. The student stands at point P, an arbitrary distance (not too close) from a flagpole as shown in Figure 2. In this problem P is at a distance of 50 feet from the flagpole.

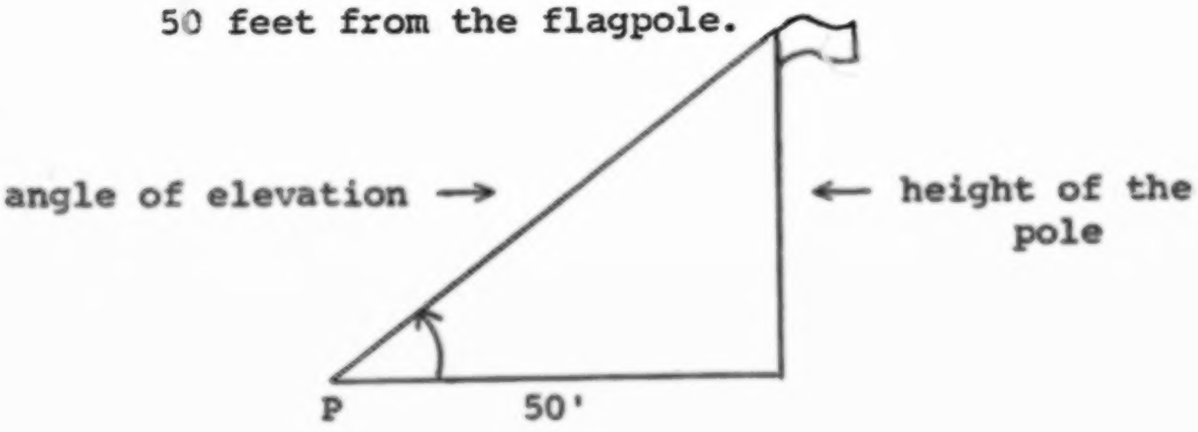

angle of elevation
height of the pole
P
50'

Figure 2

   b. Standing at point P, the student sights along the edge of the instrument to the top of the flagpole.

   c. The string will remain in a vertical position because of the attached weight and will cross the protractor at a point that represents the measure of the angle of elevation. Figure 3 shows the position for a reading of an angle of elevation of 20°.

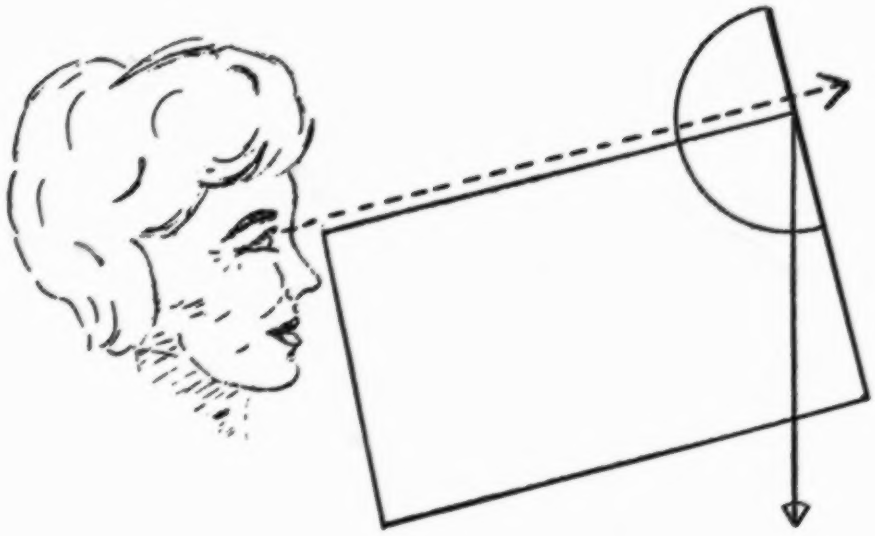


Figure 3

The following is a diagram of the scene.

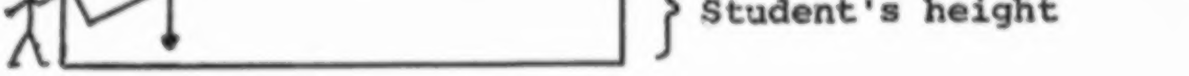
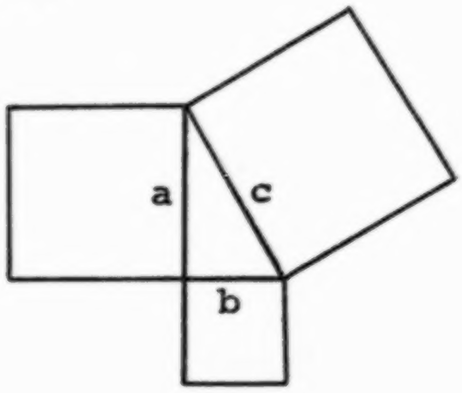

Line of sight →
x (height of flagpole)
Student's height

Figure 4

Solve the problem by the use of a scale drawing. Graph paper is ideal for this purpose. Note that construction of a scale drawing will give the height of the flagpole (x on Figure 4) above the student's horizontal line of sight. Remember to have the student add his eye level height to the height of x to get the final answer.

The Pythagorean Theorem

Many students are acquainted with the geometric interpretation of the Pythagorean Theorem, "the sum of the squares on the legs of a right triangle is equal to the square on the hypotenuse."

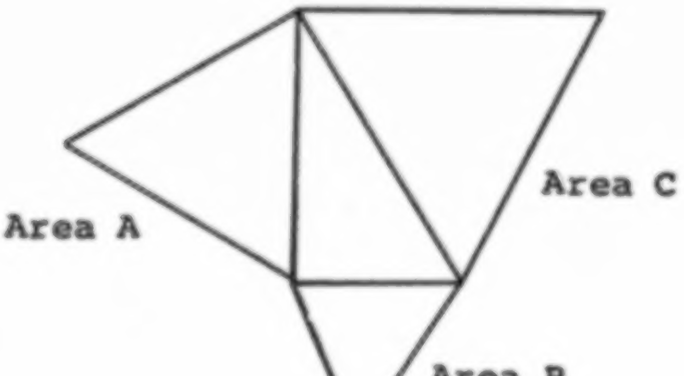


$$c^2 = a^2 + b^2$$

An interesting fact that can be proved by students is that the same area relationship holds for other figures placed on these sides as long as the three figures are similar to each other. (Reason: Areas of any two similar figures are proportional to the squares of a corresponding dimension.)

Example:

Construct equilateral triangles on the three sides of a right triangle.



(1) $\frac{\text{Area A}}{\text{Area C}} = \frac{a^2}{2}$ & $\frac{\text{Area B}}{\text{Area C}} = \frac{b^2}{2}$

(2) $a^2 + b^2 = c^2$; dividing by c^2 gives $\frac{a^2}{b^2} + \frac{b^2}{c^2} = 1$

(3) $\frac{\text{Area A}}{\text{Area C}} + \frac{\text{Area B}}{\text{Area C}} = 1$; clear fractions and get

Area A + Area B = Area C. (The equilateral triangle on the hypotenuse of a right triangle is equal to the sum of the equilateral triangles on the legs.)

Other interesting area problems in which you can prove that Area A + Area B = Area C are:

(1) Semicircles
(2) Regular hexagons
(3) Regular octagons
(4) Isosceles triangles with base angles of 75 degrees.
(5) Similar triangles
(6) Quarter circles
(7) Arches
(8) Circles

Short Cut Involving Pythagorean Theorem

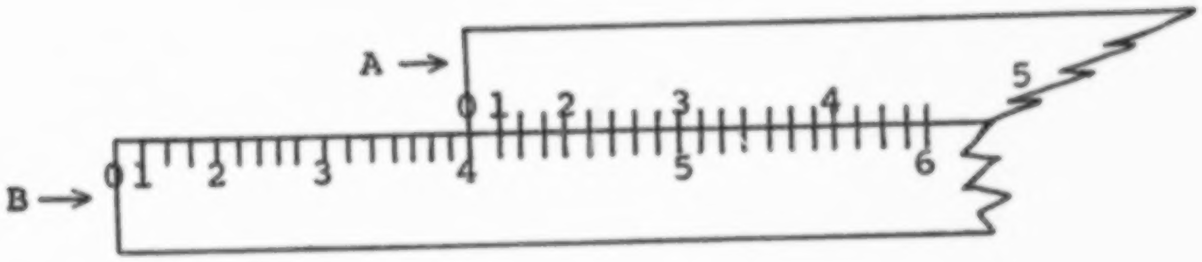
A special slide rule using graph paper can be made to solve the Pythagorean Theorem.

Construction:

1. Use two strips of graph paper (tenths to the inch). It is advisable to work with a minimum of a 10-inch strip.
2. Both scales A and B are labeled identically.
   a. Mark the beginning of the scales with a zero.
   b. Mark a 1 at the first tenth position to the right of the 0. This mark represents 1^2.
   c. Mark a 2 at the 4-tenth position to represent 2^2.

d. The 3 is placed at the 9-tenth position to represent 3^2.

e. Continue this pattern.



Use of the slide rule

The figure shows the correct settings for the solution for the hypotenuse of a right triangle having sides of 3 units and 4 units. The answer 5 is obtained by adding the scale lengths of 3 and 4.

1. Set the 0 reading of scale A over the 4 mark of scale B.
2. Find the 3-mark on scale A and under it, on scale B, read the answer 5.

The same settings of the slide rule show how to find the square root of $5^2 - 3^2$ by subtracting length 3 from length 5. The 3-mark of scale A is lined up with the 5-mark on scale B. The answer 4 is found under the 0 mark of scale A.

Suggested exercises using the slide rule:

	a	b	c
1.	5	2	
2.	2		5
3.		4	7
4.	4		8
5.	3	7	
6.	5	5	
7.		1	10
8.	6	8	

Constructions in Geometry

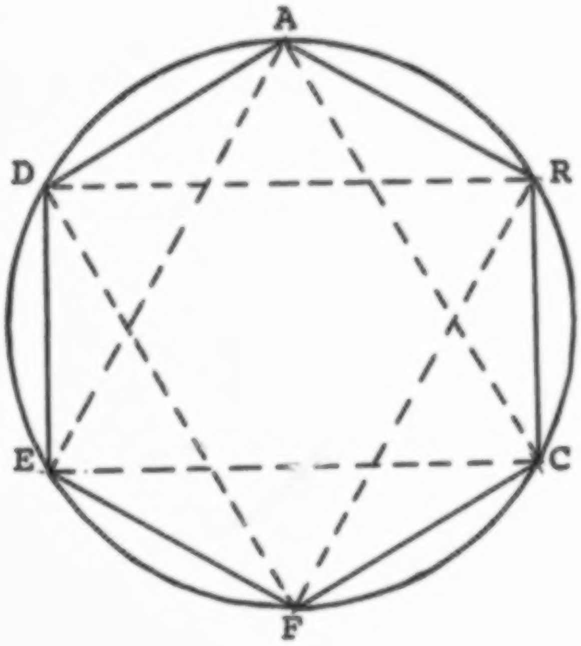
Most students find construction work in geometry interesting and exciting. There are six basic constructions using a compass and straight-edge:

1. To duplicate a given line segment.
2. To bisect a given line segment.
3. To construct an angle equal to a given angle.
4. To bisect a given angle.
5. To construct a parallel to a given line.
6. To construct a perpendicular to a line from a given point (point may be on or off the line).

Regular Hexagons

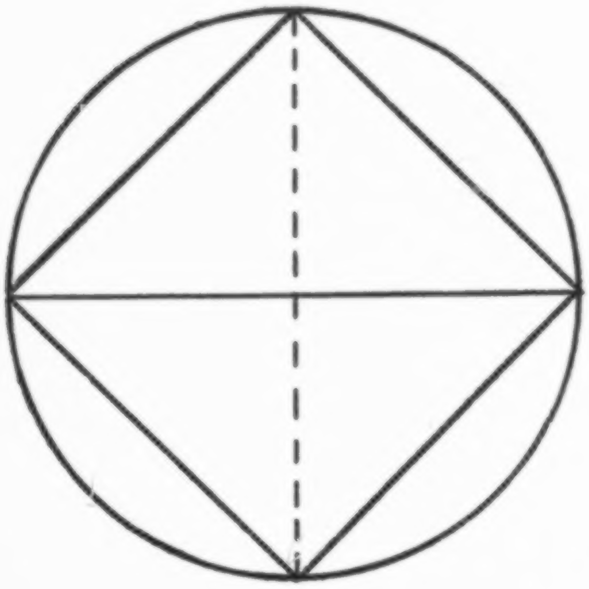
After learning to construct a regular hexagon in a circle it is a simple matter to make regular polygons with sides of 3, 12, 24, 48 . . . by bisecting a side and its corresponding arc of the polygon.

For the hexagon use r as radius for a circle and using this same radius, mark off points around the circumference of the circle. By connecting the consecutive points, a regular hexagon is formed. If every other point is connected, an equilateral triangle is formed. By drawing two triangles, ACE and DRF, a six-pointed star is formed. Many designs can be created by students from the basic regular hexagon.



Squares

Construct the perpendicular bisector of the diameter of a circle. Connect the four consecutive points on the circumference and have a perfect square. By bisecting a side of the square and its corresponding arc, a regular octagon can be constructed. Continuing in this way regular polygons of 8, 16, 32, 64 . . . sides can be constructed.

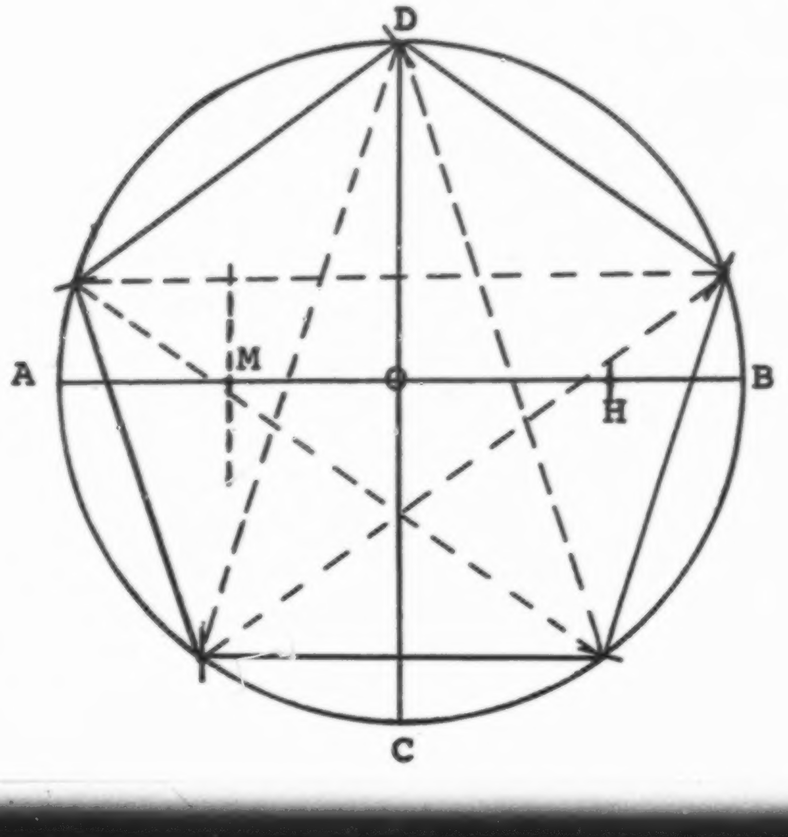


Many students want to know how to construct a regular polygon of 5, 7, 9 . . . sides.

Regular Pentagon (5 pointed star)

Construction:

1. First draw a circle and 2 perpendicular lines through the center.
2. Find the midpoint M of OA.
3. With M as center and MD as radius, draw an arc that cuts OB at H.
4. Line segment DH is the same length as a side of the regular pentagon which fits inside a circle with center O and AO as radius.
5. By connecting consecutive points on the circumference, a regular pentagon is constructed.
6. By connecting every other point, a 5 pointed star is formed.

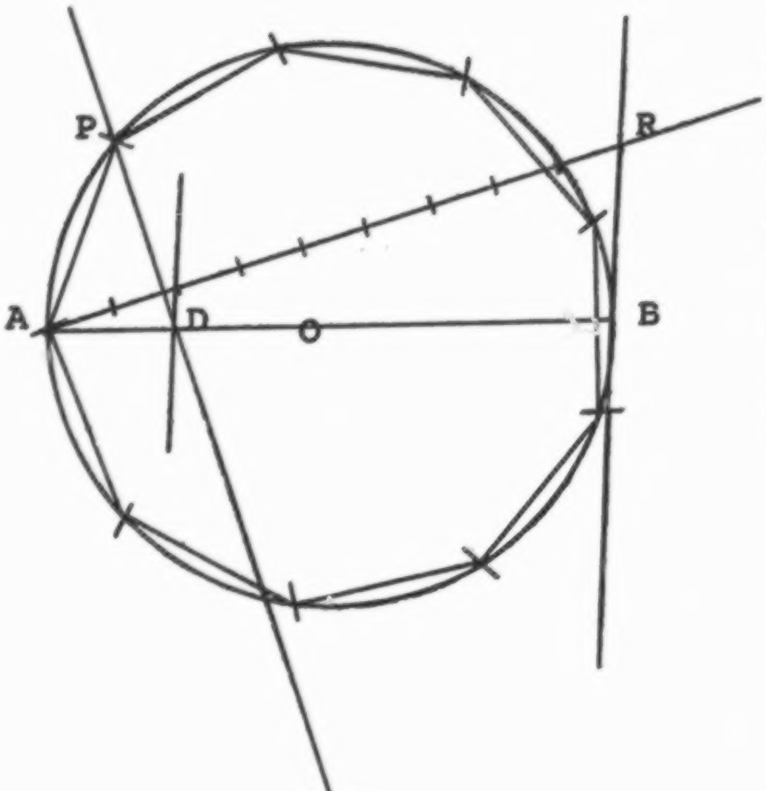


Miscellaneous Regular Polygons (Approximations)

Nonagon:

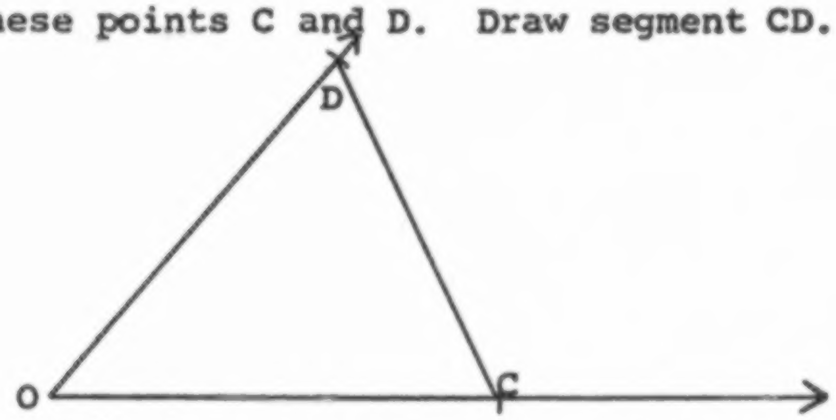
1. Draw diameter AB in circle O.
2. Draw a work line from point A.
3. Divide work line AR into 9 equal segments.
4. Connect R and B.
5. Through the #2 division mark on AR, construct a line parallel to RB and intersecting AB at point D.
6. Using AB as radius and using points A and B for the needle of the compass, locate point M.
7. Draw a line through M and D intersecting the circle at P.
8. Use line segment AP to mark off 9 equal arcs on the circumference.

This method will give close approximations for regular polygons of 20 or less sides. It is true that time would be saved by the teacher making all geometric drawings on the board, but nothing quite takes the place of the pupil's own drawings made at his desk. Let him have the fun of discovering geometric facts for himself.



Sample Exercises to Discover Geometric Concepts

1. From point O, draw two rays forming an angle of 50°. On each ray locate a point 2 inches from O. Letter these points C and D. Draw segment CD.



   a. What appears to be true about the angles CDO and DCO?

   b. Check your judgment by measuring the angles.

2. On a line AB, at point P, draw a line PC making APC = 80°. Measure CPB. (Do similar exercises several times to discover what is true about supplementary angles.) What kind of angles are APC and CPB?

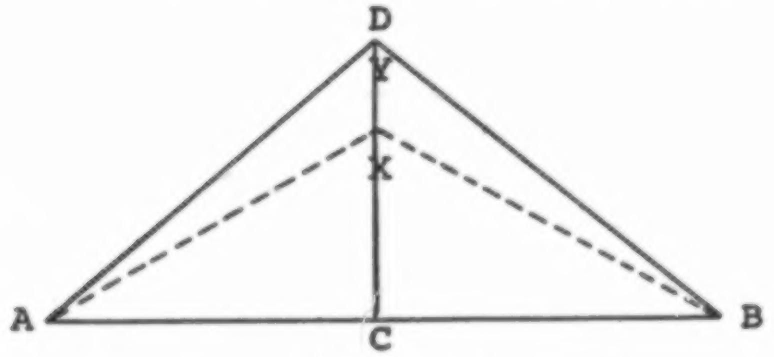
3. Draw a segment AB, 3 inches long. Place point C at its center. At C, draw a perpendicular CD to AB above AB.

   a. Take point X on this perpendicular and connect it with A and B. How does AX appear to compare with BX? Justify.

   b. Repeat part "a" by placing point Y somewhere else on CD.

c. Suppose a point Z on CD were 10 inches from A. How far would it be from B?

d. Try to express in a sentence the fact you have discovered in parts a, b, and c.



(In figure for this problem, all angles such as XAB and XBA can be compared and another fact discovered.)

4. On a sheet of paper locate:

   a. Three points not on the same straight line.

   b. Four points, no three of which are on the same straight line.

   c. Five points, no three of which are on the same straight line.

   d. And so on.

By taking two points for each line, what is the largest number of lines that can be drawn through these points? Arrange answers in form of a table.

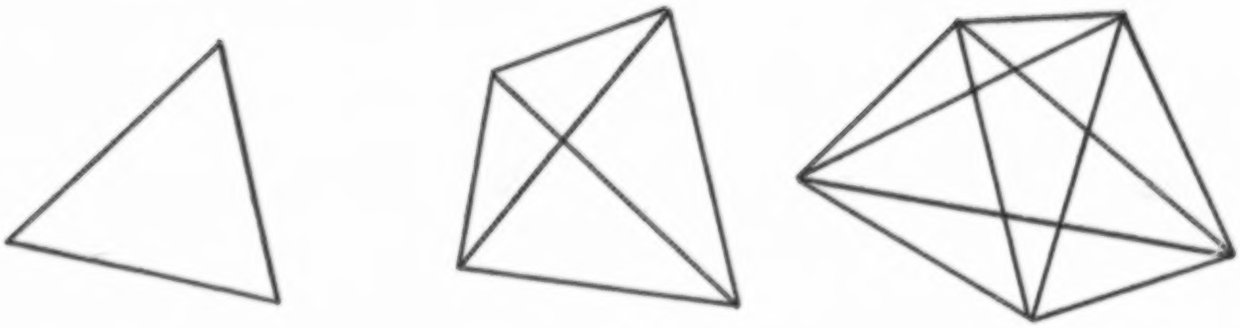


Table:

Points	3	4	5	6	7	8	9	. . . p
Lines	3	6	...	...	...	...	...	 L

Try to discover a formula based on this table giving the relation among any number of points (p) and the largest number of lines (L) determined by them.

Formula: $L = p + p(\frac{p-3}{2})$

5. How many degrees are there in the sum of the measures of the interior angles of each of these polygons?

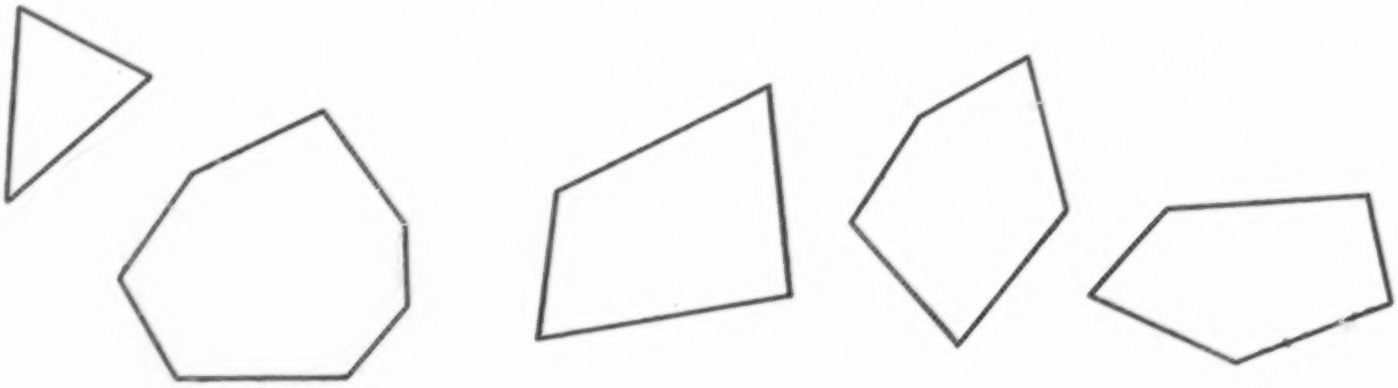


Table:

Number of Sides	Sum of Measures
3	180
4	360
5	
6	
7	
.	
.	
n	

Try to discover a formula that will give the relation between the sum of the angle measures of any polygon and its number of sides. [S = (n - 2) 180].

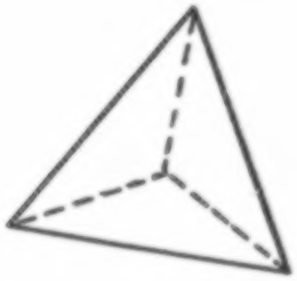
Similar worksheets and tables can be used to let the students discover the formulae for the number of diagonals from a single vertex, the number of triangular regions into which a polygon can be divided by diagonals from a single vertex, and the total number of different diagonals that can be drawn in any polygon.

Spatial Relationships

To aid in developing spatial perception in geometry, exercises of this type may help (build models).

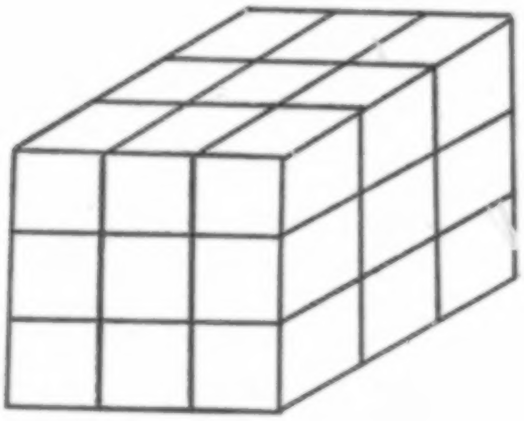
1. Using six popsicle sticks of equal length, how can they be arranged so that four triangles are formed whose sides are the length of the popsicle sticks.

   Answer:



(Tetrahedron)

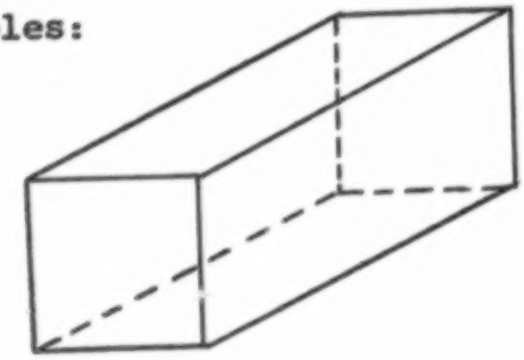
2. Ask the students to visualize a 3-inch cube that is painted red. Assume that this cube is cut into 27 one-inch cubes as in the figure.



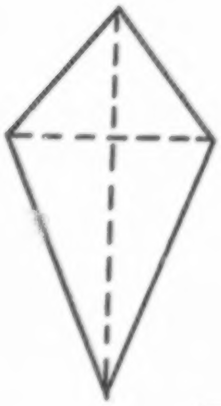
How many of these one-inch cubes will have red paint on none of their faces? on one face only? on two faces only? On more than three faces? Building blocks or sugar cubes are two things that can be used to build a model if the students cannot picture the original cube.

3. Students learn many geometric facts by constructing models of geometric solids. Students can draw patterns for these models on graph paper. The teacher may have the patterns prepared in advance. An overhead projector can be put to good use with transparencies of patterns for solids. By constructing models of geometric solids students may be led to discover Euler's formula V + F = E + 2. This states that the number of vertices (V) plus the number of faces (F) is equal to two more than the number of edges (E).

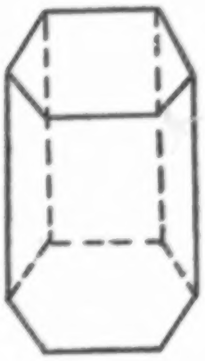
Examples:



Rectangular Prism:
V = 8
F = 6
E = 12

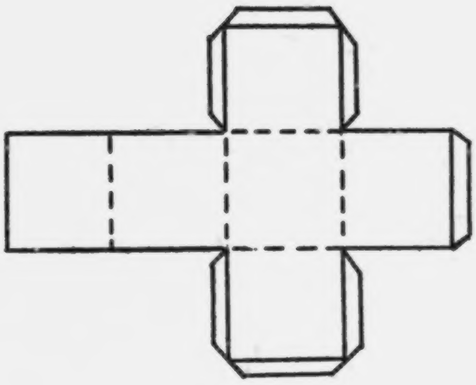


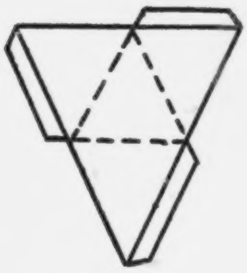
Tetrahedron:
V = 4
F = 4
E = 6

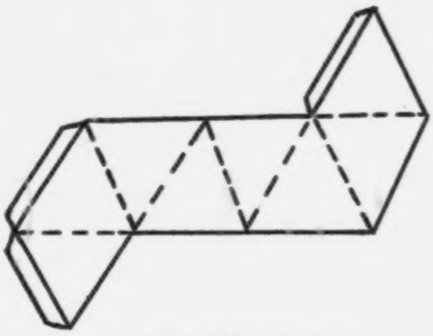


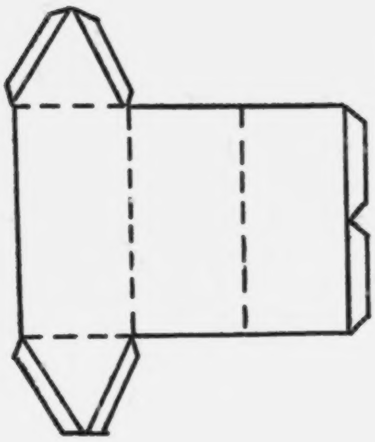
Hexagonal Prism:
V = 12
F = 8
E = 18

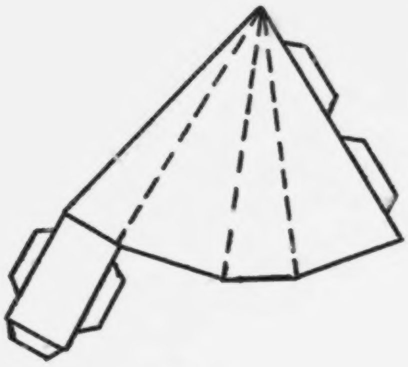
Sample Patterns for Models

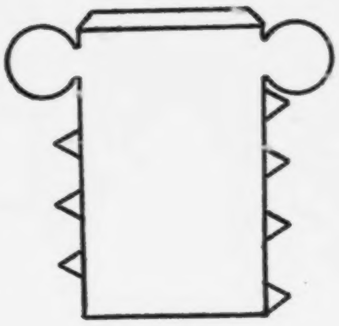

Cube

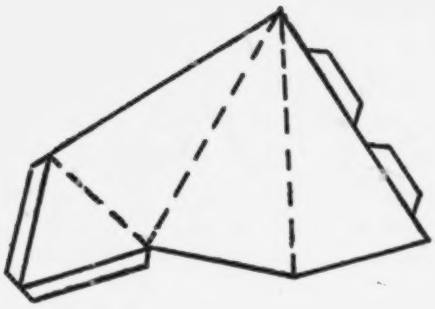

Tetrahedron

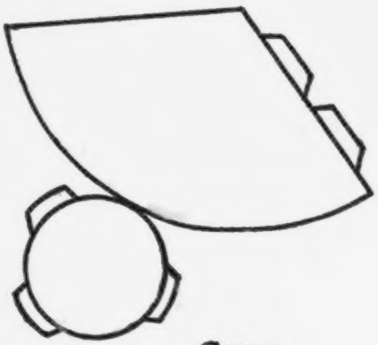

Octahedron

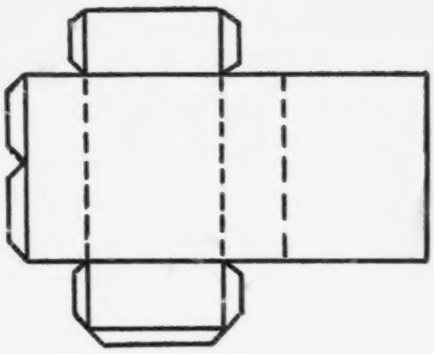

Triangular Prism


Rectangular Pyramid


Cylinder


Triangular Pyramid


Cone


Rectangular Prism

Area of a Circle

A. The discovery approach of "cut-and-make-a-familiar-figure" is often overlooked by the teacher because of lack of equipment available for student use. However, this method for finding the area of a circle can be demonstrated very easily by use of a flannel board or an overhead projector if student equipment is lacking.

A circular region cut into fourths can be reassembled into a region that resembles a parallelogram as in Figure 1. The height might be the radius of the circle, but the "base" does not resemble anything at this stage.

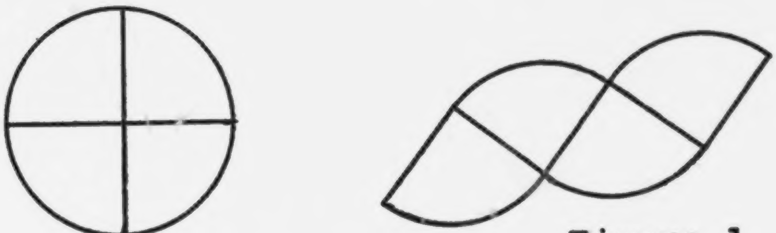


Figure 1

When the circular region is divided successively into smaller and smaller subdivisions and reassembled, it will better approximate the shape of a parallelogram. The height of the parallelogram seems to be the radius of the circle and the base of the parallelogram seems to be one-half of the circumference of the circle. Thus, the circular region cut into 16ths and reassembled would look like Figure 2 (page 124).

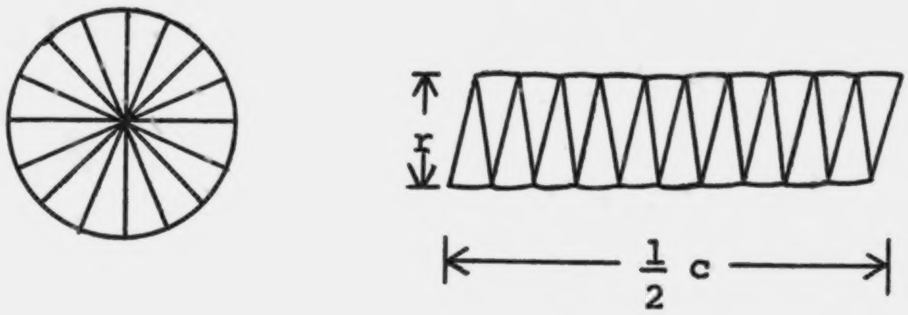


Figure 2

Since the area of a parallelogram is found by multiplying the base times the height and can be expressed by the formula $A = bh$, then by substitution this formula is expressed as $A = \frac{1}{2}\,cr$. Since the circumference of a circle is $c = 2\pi r$, then by substitution this formula becomes $A = \frac{1}{2} \times 2\pi r \times r$; when simplified this becomes $A = \pi r^2$.

B. Another method to discover the area of a circular region.

Necessary materials:

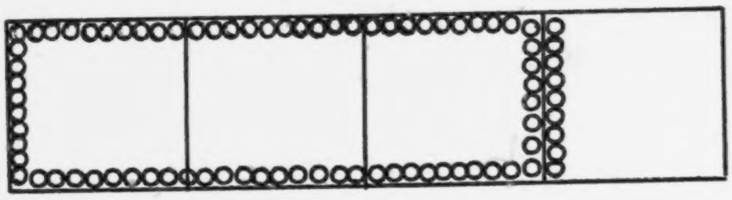
1. Circular plastic lid off a container (such as a coffee can).
2. Four squares having sides equal to the radius of the circular lid.
3. Beads or marbles.

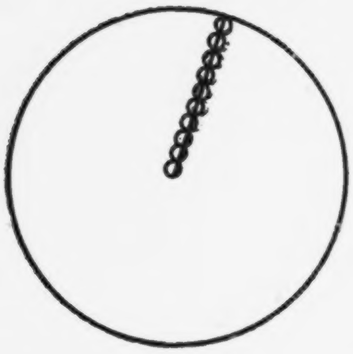
Procedure:

If overhead projector is available, place the plastic lid on the projector and fill it with beads. Empty these beads into the prepared squares; they will completely fill three of the squares and approximately $\frac{1}{7}$ of the fourth square. Therefore, the area of the circular region is approximately $3\frac{1}{7}$ radius squared regions. Expressed as a formula this is:

$$A = 3\frac{1}{7}r^2 \quad .$$

This demonstration can be done on a desk top if an overhead projector is not available.





Preparation of Lesson Plans

A teacher of mathematics must be prepared to meet all problems that arise in his classes. Each pupil is an individual being with varying abilities. To be able to help each student work toward his potential, the teacher must be prepared himself through the guidance of a good lesson plan. What are some of the ideas that should go into a lesson plan?

A. Objective or purpose of a lesson is clearly stated.

B. Recall or review.

  1. Includes written or oral drill.
  2. Relates to current unit.
  3. Checks drill.
  4. Makes smooth transition to next phase of lesson.

C. Motivation.

  1. Uses puzzle problems or interest arouser.
  2. Draw from pupil's experiences.
  3. Includes current topics (news, sports, space flights, etc.)
  4. Stresses importance of topic to be presented.
  5. Illustrates with models, devices, etc.

D. Presentation.

  1. Gives a clear explanation and development.
  2. Involves student participation.
  3. Phrases questions to stimulate thinking.

   4. Explains why as well as how.
   5. Uses illustrative material--visual aids.
   6. Encourages students to ask questions.
   7. Uses a variety of approaches.

E. Assignment.
   1. Relates to lesson.
   2. Varies according to student ability.

F. Summary.
   1. Ties together loose ends.
   2. Restates important concepts and skills.
   3. Shows evidence of accomplishment of objective.

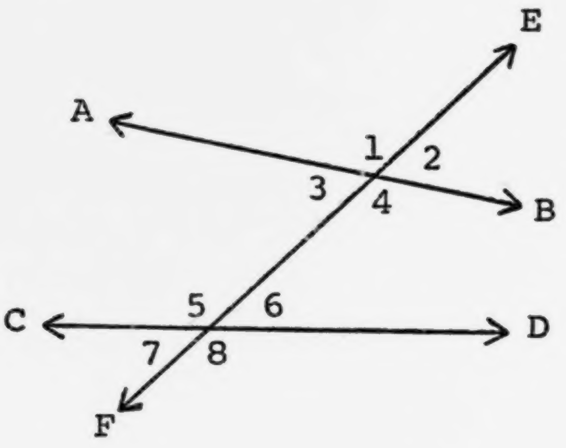
Sample Lesson Plan

A. Objective:

To prove that if two lines are intersected by a transversal forming equal alternate interior angles, then the lines are parallel.

B. Recall:

1.



Given: Two lines AB and CD cut by transversal EF.

Name:

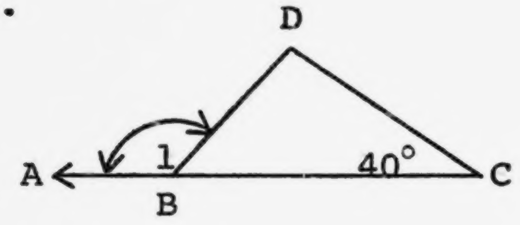
a. 2 pairs alt. interior angles.

b. 2 pairs vertical angles.

c. 2 pairs alt. exterior angles.

d. 2 pairs corresponding angles.

2.



$\angle 1$ is

a. $= 40^\circ$

b. $< 40^\circ$

c. $> 40^\circ$

During the recall have homework problems worked at the board.

Check homework and recall problems.

Collect homework.

C. Motivation and Presentation:

(Have statement and figure for theorem to be proved on the blackboard.)

Indirect Proof: Sometimes it is hard to prove that a condition is true directly, but if we can prove a false conclusion to be absurd, then the only possibility remaining is for the true condition to be correct.

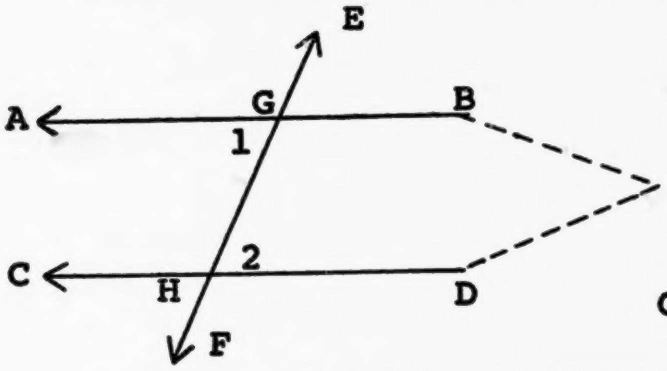
Examples:

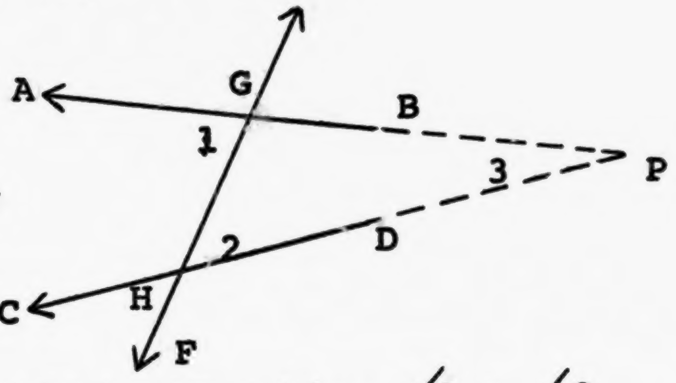
1. a. Every integer is either odd or even.
    b. The integer n is not odd.
    c. Conclusion: ____________________
2. a. Segment AB is greater than, equal to, or less than XY.
    b. AB is not greater than XY, nor equal to XY.
    c. Conclusion: ____________________
3. What two possibilities exist for two lines in a plane?
    a. Parallel lines.
    b. Intersecting lines (non-parallel).

Theorem:

If two lines are intersected by a transversal forming equal alternate interior angles, then the lines are parallel.

(NOTE: Have students suggest steps after deciding upon a plan of proof or analysis of the problem. Let students explain a reason for each suggested step.)





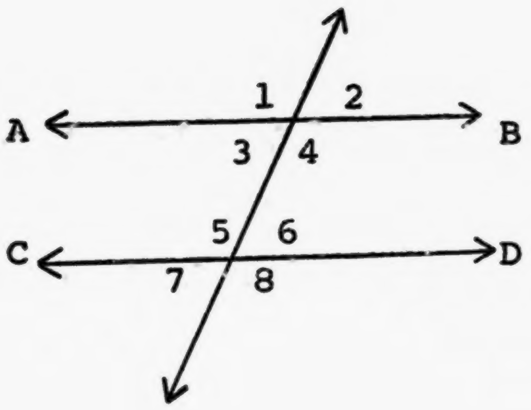
Given: AB and CD intersected by EF making $\angle 1 = \angle 2$.

Prove: AB parallel to CD.

Analysis: Lines in a plane must be parallel or intersect. Prove that the lines can't intersect; therefore, they must be parallel.

Steps	Reasons
1. If AB is not parallel to CD, lines must meet at some point P and thus form with GH the $\triangle$ GPH.	1. Two non-parallel lines will meet if produced far enough.
2. $\angle 1 > \angle 3$	2. An exterior angle of a triangle is greater than either remote interior angle.
3. But $\angle 1$ cannot be $> \angle 3$.	3. $\angle 1 = \angle 2$ by the given statement.
4. Therefore AB $\|\|$ CD	4. Since AB and CD cannot meet, they must be parallel.

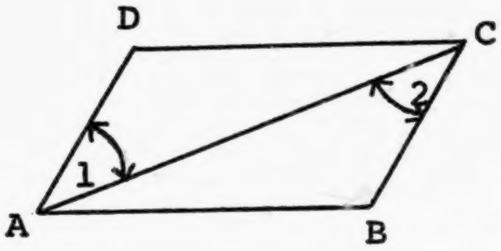
D. Applied Problems:



Is AB || CD if:

1. $\angle 3 = 50^\circ$ and $\angle 6 = 50^\circ$?
2. $\angle 4 = 100^\circ$ and $\angle 5 = 102^\circ$?
3. $\angle 4 = 100^\circ$ and $\angle 5 = 100^\circ$?
4. $\angle 3 = 50^\circ$ and $\angle 8 = 130^\circ$?
5. $\angle 2 = 50^\circ$ and $\angle 7 = 60^\circ$?

E. Summary:



1. Name the two lines and the transversal which form angles 1 and 2.
2. Complete the statement: Angles 1 and 2 are ___________ angles.
3. If $\angle 1 = \angle 2$, name the lines which are parallel. Give the reason. (State theorem in full).

F. Assignment: p. 71: #3, 5, 9. (For example.)

Sample Questions from Tests that Applicants Must Take to Obtain Work

These questions should suggest why students should study all types of mathematics. You may choose to use these questions as a diagnostic test.

Teachers probably could secure other sample test questions from stores, business houses, and industries in the local school area.

1. Round off the following numbers to 3 decimal places:

   .0156

   .0625

   .1875

   .5625

   .7556

2. A disc is 20 inches in diameter. What is its area?

   $A = \pi R^2$ $\pi = 3.1416$

   If the disc is 2 inches thick, what is its volume?

   $V = Ah$

   If there is a 10-inch diameter hole in the disc and the material weighs .1 lbs. per cubic inch, what is its weight?

3. A soldier shooting at a target hits it $12\frac{1}{2}\%$ of the time. How many times must he shoot to be certain he will register 100 hits?

4. Look at the row of numbers below. What number should come next?

   73 66 59 52 45 38 _?_

5. Five pounds of meat sells for \$2.00; how many pounds can you buy for 80 cents?

6. Two men caught 36 fish; one caught 8 times as many as the other. How many fish did each man catch?

7. An automobile that costs \$2490 has decreased in value by $33\frac{1}{3}\%$ by the end of the season. What was its value at that time?

8. One number in the following series does not fit in with the pattern set by the others. What should that number be?

   5 6 9 10 13 14 15

9. Which number in the following group of numbers represents the smallest amount?

   9 2 1 .7 .77 .99

10. A train travels 70 feet in $\frac{1}{10}$ of a second. At this same speed, how many feet will it travel in $3\frac{1}{2}$ seconds?

11. Three men form a partnership and agree to divide the profits equally. X invests \$6500, Y invests \$2000, and Z invests \$1500. If the profits are \$3000, how much less does X receive than if the profits were divided in proportion to the amount invested?

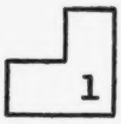
12. If $3\frac{1}{2}$ tons of coal cost \$35, what will $4\frac{1}{2}$ tons cost?

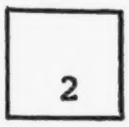
13. When the price of gasoline increased from 16.4 cents to 20.5 cents, what was the percent increase in cost of gasoline?

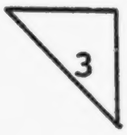
14. A dealer bought some cars for \$2500. He sold them for \$2900, making \$50 on each car. How many cars were involved?

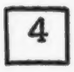
15. A mechanic works 40 hours in one week at $\$2.69\frac{1}{2}$ per hour. How much pay does he receive for one week's work?

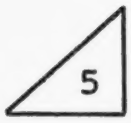
16. Four of the following 5 parts can be fitted together in such a way as to make a triangle. Which 4 are they?



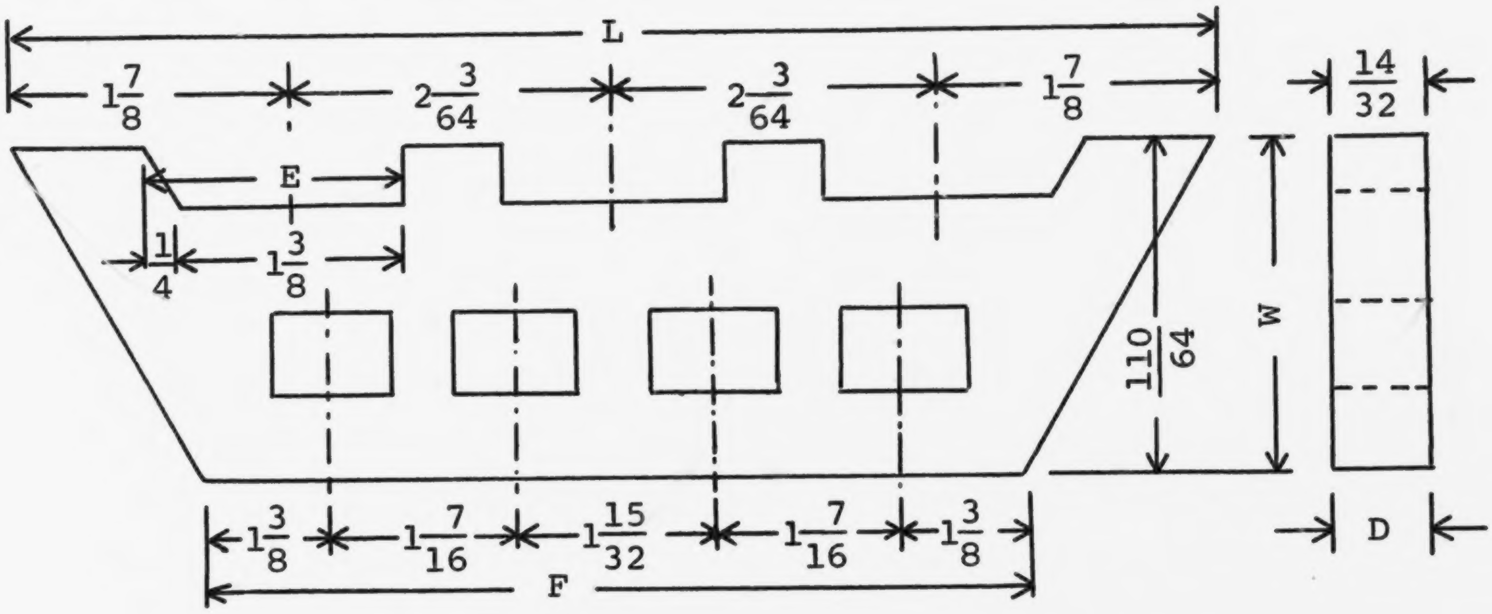








17. Using this figure, answer the following questions:


L
$1\frac{7}{8}$
$2\frac{3}{64}$
$2\frac{3}{64}$
$1\frac{7}{8}$
$\frac{14}{32}$
E
$\frac{1}{4}$
$1\frac{3}{8}$
$\frac{110}{64}$
W
$1\frac{3}{8}$
$1\frac{7}{16}$
$1\frac{15}{32}$
$1\frac{7}{16}$
$1\frac{3}{8}$
F
D

a. Express the thickness dimension (D) as a fraction in its simplest form.

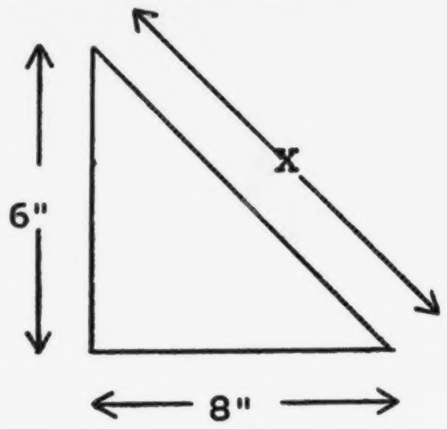
b. What is the distance (E)? (Reduce fraction if necessary.)

c. What is the width (W)? (Reduce fraction if necessary.)

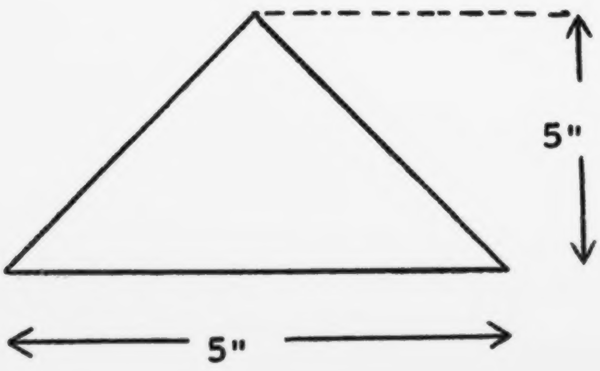
d. What is the length (F)? (Reduce fraction if necessary.)

e. What is the length (L)? (Reduce fraction if necessary.)

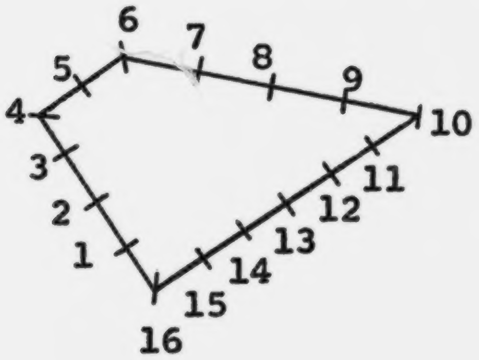
18. Find the length of line "X."



19. Find the area of this triangle.



20. This geometric figure can be divided by a straight line into two parts which will fit together in a certain way to make a perfect square. Draw such a line by joining 2 of the numbers. Then write these numbers as the answer.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Suggested Improvements

Page No.	Comment or Correction

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